129A Lecture Notes
Notes on Special Relativity

1 Why Relativity?

Particle Physics aims to study structure of space, time and matter at its most fundamental level. It necessarily means that we study physics at the shortest distance scales as possible. To probe short distances, Heisenberg's uncertainty principle $\Delta x \Delta p \geq \hbar/2$ states that we need to provide large momentum. In other words, we are looking at scattering phenomena of particles with large momenta and hence high energy. In practice, it often means the use of particle accelerators that can bombard projectile particles (beams) on target (or opposing beams) at very high momenta or energies. This is why terms "particle physics" and "high-energy physics" had been often used interchangingly.[1] As we increase momentum of the particle, eventually it exceeds the mass of the particle times the speed of light mc, and the particle becomes relativistic. Therefore, special relativity is an integral part of particle physics and we need to understand it thoroughly. Even though special relativity has been discussed already in 110AB, I'd like to review it in the way that I can use it later on.

Relativity itself of course is one of the major revolutions in physics in 20th century, together with quantum mechanics. The way Einstein discovered it was based on theoretical consideration that Maxwell's equation had unusual invariance, Lorentz invariance. He boldly stated that it must be the invariance of our space and time. It completely changed the way we view our space and time, so intertwined that it is now called "spacetime," leading to exotic phenomena such as time dilation and Lorentz contraction. Its marriage with quantum mechanics further led to a dramatic prediction of the existence of anti-matter.

[1] Of course, particle physics experiments that do not use accelerators had also been quite successful as we will discuss in this course, such as neutrino physics and cosmology, and sometimes people distinguish these two terms.

2 Gallilean space and time

In the pre-relativity view of the space and time, different reference frames are related to each other by Gallilean transformations. The notion of "relativity" itself was already there. Namely, laws of physics remain the same in any "inertial" reference frames that move relative to each other at constant speed.[2]

Suppose the reference frame B is moving relative to the reference frame A by velocity $\vec{V}$. Their origins coincide at $t = 0$. If a particle moves at the position $\vec{x}(t)$ in the frame A, its position is simply given by

$$\vec{x}'(t) = \vec{x}(t) - \vec{V}t. \tag{1}$$

This coordinate change is called Gallilean transformation. Just by taking the time derivative of both sides, the velocities are related by

$$\vec{v}' \equiv \dot{\vec{x}}' = \dot{\vec{x}} - \vec{V} = \vec{v} - \vec{V}. \tag{2}$$

Namely, velocities in two frames are related by a simple vector sum.

There is an implicit assumption in these equations that the time is common to both frames,

$$t' = t \tag{3}$$

It is normally not stated explicitly in discussions of mechanics, and is taken for granted. No matter which frame you are in, time just flows the same, right? This naive statement of course underwent big revision under relativity.

3 Michaelson–Morley experiment

Here, we follow a bottom-up approach to introduce relativity unlike the original Einstein's argument, namely starting with an experimental evidence.

Michaelson and Morley, using their sensitive interferometer mounted on hills, demonstrated that the speed of light c is a fundamental constant independent of the direction and the motion of the Earth itself.[3] It implies that

[2] Of course accelerating frames exhibit different laws of physics that include the centrifugal force and Corioli's force.

[3] There were looking for the evidence of "ether", a postulated medium that transmits electromagenetic waves. People in those days could not accept the fact that waves can

change from one reference frame to another must be done in such a way that the speed of light remains the same.

This experimental fact goes against Eq. (2), and hence the Gallilean transformation itself. What we have to do is to change the relationship (coordinate transformation) between two frames of reference. We try to figure out what that is by requiring that the speed of light remains the same.

Consider small time and space intervals dt and $d\vec{x} = (dx, dy, dz)$ in the frame A. What we would like to figure out is what the corresponding intervals are in the frame B, dt' and $d\vec{x}'$. For the light, the time and space intervals are related by its speed,

$$(d\vec{x})^2 = (dx)^2 + (dy)^2 + (dz)^2 = (cdt)^2. \tag{4}$$

The fact that the speed of light is common to both frames means this is also true in the frame B,

$$(d\vec{x}')^2 = (dx')^2 + (dy')^2 + (dz')^2 = (cdt')^2. \tag{5}$$

In Gallilean transformation Eq. (1,3), the intervals in two reference frames are given by

$$dt' = dt, \tag{6}$$

$$d\vec{x}' = d\vec{x} - \vec{V}dt. \tag{7}$$

If you insert these expressions into Eq. (4), you find

$$(d\vec{x}')^2 = (d\vec{x})^2 - 2\vec{V} \cdot d\vec{x}dt + \vec{V}^2(dt)^2 = (c^2 + \vec{V}^2)(dt)^2 - 2\vec{V} \cdot d\vec{x}dt. \tag{8}$$

Therefore Eq. (5) is not satisfied.

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propagate in "vacuum". All other waves people were familiar with had some kind of medium. Sound through air, ocean waves through water, etc. "Ether" was assumed to be weightless and frictionless as not to disturb the planetary motion over billions of years. If there existed "ether" as the medium of the electromagnetic wave, there must be its "rest frame." In any reference frame that moves relative to the rest frame, the speed of light must be different. By measuring the speed of light along with and orthogonal to the motion of the Earth, their experiment was sensitive enough to demonstrate this point. This expectation was wonderfully crushed.

4 Lorentz Transformation

We are forced to look for coordinate transformations that preserve Eqs. (4,5). The only assumption we make is that the transformation is *linear*, *i.e.*, it is a matrix multiplication on coordinates.[4] Because it is awkward to deal with all three spatial coordinates all the time, let us focus on z direction only for the moment. We will generalize the result to the case of three spatial coodinates later on. We use cdt and dz so that both time and space coordinates have the same dimension. The matrix multiplication is simply

$$\begin{pmatrix} cdt' \\ dz' \end{pmatrix} = A(V) \begin{pmatrix} cdt \\ dz \end{pmatrix} = \begin{pmatrix} A_{tt}(V) & A_{tz}(V) \\ A_{zt}(V) & A_{zz}(V) \end{pmatrix} \begin{pmatrix} cdt \\ dz \end{pmatrix}. \tag{9}$$

The Gallilean transformation Eq. (7) corresponds to the matrix

$$A(V) = \begin{pmatrix} 1 & -\frac{V}{c} \\ 0 & 1 \end{pmatrix}. \tag{10}$$

We start with requirements that do not concern with speed of light. First obvious requirement is that an object at rest in frame A must appear to move with velocity $-V$ in frame B. In this case, $dz = 0$ in frame A because it is not moving. In frame B, $dz'/dt' = -V$. Therefore,

$$-\frac{V}{c} = \frac{dz'}{cdt'} = \frac{A_{zt}cdt}{A_{tt}cdt} = \frac{A_{zt}}{A_{tt}}. \tag{11}$$

The next requirement is less obvious. If you go from frame A to B with relative velocity V, and then combe back to A, which is another coordinate transformation with relative velocity $-V$, you haven't done anything. Therefore, the matrices $A(V)$ and $A(-V)$ must be inverse of each other. On the other hand, the coordinate transformation $A(-V)$ can be done by flipping the z direction, doing the transformation $A(V)$, and then flipping the z direction back. In other words,

$$A(-V) = PA(V)P, \tag{12}$$

where P flips the z coordinate,

$$P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{13}$$

[4] Non-linear transformations appear in the general theory of relativity, Einstein's theory of gravity. Such transformations are called general coordinate transformations.

Therefore, the requirement is that $A(-V)A(V) = PA(V)PA(V) = I$, where I is the unit matrix. By writing it out explicitly, we find

$$PA(V)PA(V) = \begin{pmatrix} A_{tt}^2 - A_{tz}A_{zt} & A_{tz}(A_{tt} - A_{zz}) \\ A_{zt}(-A_{tt} + A_{zz}) & -A_{zt}A_{tz} + A_{zz}^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (14)$$

The off-diagonal elements must vanish, and hence $A_{tt} = A_{zz}$. Then the diagonal elements give the constraint $A_{tt}^2 - A_{tz}A_{zt} = 1$. Together with Eq. (11), we can solve it up to a single unknown parameter A_{tt},

$$A(V) = A_{tt}(V) \begin{pmatrix} 1 & -\frac{c}{V}\frac{A_{tt}^2-1}{A_{tt}^2} \\ -\frac{V}{c} & 1 \end{pmatrix}. \quad (15)$$

At this point, it is easy to see that Gallilean transformation Eq. (10) is a special case of this general form with $A_{tt} = 1$. In fact, if you *require* that "time" must be common to both frames, you need $A_{tt} = 1$, and Eq. (10) is obtained as the unique choice under this requirement. Because "daily" phenomena appear to support the idea that "time" is common no matter how fast you go, people implicitly made this requirement and the Gallilean transformation followed as the only possibility.

Now comes the crucial requirement that the speed of light is the same in both frames. The requirement is that $(cdt')^2 - (dz')^2 = 0$ if $(cdt)^2 - (dz)^2 = 0$. We subsitute dt' and dz' into Eq. (5) and find for light going right $cdt = dz$,

$$\begin{aligned} 0 &= (cdt')^2 - (dz')^2 \\ &= (A_{tt}cdt + A_{tz}dz)^2 - (A_{zt}cdt + A_{zz}dz)^2 \\ &= ((A_{tt} + A_{tz})^2 - (A_{zt} + A_{zz})^2)(dz)^2. \end{aligned} \quad (16)$$

This combination must vanish. Using the result obtained in Eq. (15), we find

$$A_{tt}^2 \left(\left(1 - \frac{c}{V}\frac{A_{tt}^2 - 1}{A_{tt}^2}\right)^2 - \left(1 - \frac{V}{c}\right)^2 \right) = 0. \quad (17)$$

Therefore,

$$\frac{A_{tt}^2 - 1}{A_{tt}^2} = \frac{V^2}{c^2}, \quad (18)$$

and hence

$$A_{tt} = \frac{1}{\sqrt{1 - V^2/c^2}}. \quad (19)$$

We have now completely determined what coordinate transformation is consistent with Michaelson–Morley experiment. As you have seen in the above discussion, this is the unique choice. This is the Lorentz transformation that allows us to take physical quantities from one reference frame to another.

It is conventional to use the notation

$$\begin{aligned} \beta &= \frac{V}{c} < 1, & (20) \\ \gamma &= \frac{1}{\sqrt{1-\beta^2}} > 1. & (21) \end{aligned}$$

Using this notation, the Lorentz transformation is written as

$$A(V) = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix}. \tag{22}$$

This is the expression we will use *a lot* in special relativity.

When we consider all three spatial coordinates, the relative velocity between two frames is a vector $\vec{V}$, and we naturally define $\vec{\beta} = \vec{V}/c$. The definition of γ is similar, $\gamma = 1/\sqrt{1-\vec{\beta}^2}$. Using the notation that $\vec{\beta}$ is a column vector, the Lorentz transformation is given by

$$A(\vec{V}) = \left(\begin{array}{c|c} \gamma & -\gamma\vec{\beta}^T \\ \hline -\gamma\vec{\beta} & (I - \frac{\vec{\beta}\vec{\beta}^T}{\vec{\beta}^2}) + \gamma\frac{\vec{\beta}\vec{\beta}^T}{\vec{\beta}^2} \end{array} \right). \tag{23}$$

The expression looks complicated, but what is means is very simple. The lower 3×3 block has two terms. They decompose any vector into a piece that is parallel to $\vec{\beta}$ by virtue of the projection $\vec{\beta}\vec{\beta}^T/\vec{\beta}^2$, and another that is orthogonal to $\vec{\beta}$ by its complement $I - \vec{\beta}\vec{\beta}^T/\vec{\beta}^2$. The first term is not changed because the vector orthogonal to the relative velocity $\vec{V}$ is not affected, while the second term is multipled by γ.

5 Implications of Lorentz Transformations

There is one *major* difference between Gallilean transformation Eq. (10) and Lorentz transformaion Eq. (22). It is that the "time" is different in two frames after a Lorentz transformation.

Suppose you consider a time interval dt in the rest frame of an object ($dz = 0$). Then in a frame where the object is moving, you find

$$dt' = \gamma dt. \tag{24}$$

This is the famous "time dilation" effect. For instance, a muon,[5] a charged particle similar to the electron, decays with a lifetime $\tau = (2.19703 \pm 0.00004) \times 10^{-6}$ sec. At the velocity $v = c\beta$, it appears that it travels only over distance $\beta c\tau$. However, because of the time dilation effect, it survives over much longer time $\gamma\tau$, and hence the distance it can travel is $\gamma\beta c\tau$. This is why muons, produced by the reaction of cosmic rays with atmosphere at an altitude of 15–20 km, can reach the surface. You may have seen a demo where a particle detector, often a spark chamber, literally keeps showing tracks of muons going downward.

Another related point is the "Lorentz contraction." In the rest frame of the atmosphere of thickness $dz' = L$, the muon takes the time interval $dt' = L/v = L/(c\beta)$ to traverse it. In the rest frame of muon, the Lorentz transformation gives

$$\begin{pmatrix} cdt \\ dz \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} L/\beta \\ L \end{pmatrix} = \begin{pmatrix} L/(\gamma\beta) \\ 0 \end{pmatrix}. \tag{25}$$

The atmosphere of certain thickness comes rushing on top of him with velocity V and it takes the time interval $L/(\gamma\beta c)$ to pass him. Therefore he would conclude that the atmosphere had the thickness $L/(\gamma\beta c) \times V = L/\gamma$. It appears "thinner" than it is in its rest frame. In the muon rest frame, he can survive until the entire atmosphere passes him not because he lives longer but because the atmosphere is squashed. In other words, Lorentz contraction and time dilation are the same story told by different observers.

It is useful to check that Lorentz transformation reduces to Gallilean transformation for small velocities $V \ll c$. For "daily" phenomena, typical times intervals dt are measured in seconds, and typical distances dz in centimeters. Given $c = 3 \times 10^{10}\text{cm}\,\text{sec}^{-1}$, it follows that $cdt \gg dz$. Writing out Eq. (22), we find

$$dt' = \frac{1}{\sqrt{1 - V^2/c^2}} \left(dt - \frac{V}{c^2} dz \right), \tag{26}$$

$$dz' = \frac{1}{\sqrt{1 - V^2/c^2}} \left(dz - V dt \right). \tag{27}$$

[5]We will discuss this particle a lot more later in this class.

In the first line, $\frac{V}{c^2}dz$ is completely negligible compared to dt, while both dz and Vdt can be comparable in the second line. The prefactor $\gamma = 1/\sqrt{1-V^2/c^2}$ can be well approximated by 1 up to negligible corrections of $O(V^2/c^2)$. In the end the Lorentz transformation can be well approximated by

$$dt' = dt, \tag{28}$$
$$dz' = dz - Vdt. \tag{29}$$

This is nothing but the Gallilean transformation Eq. (7). In other words, Lorentz transformation indeed does reduce to Gallilean transformation at small velocities and.

Another important consequence of the Lorentz transformation is that the combination

$$(cd\tau)^2 \equiv (cdt)^2 - (dx)^2 - (dy)^2 - (dz)^2 \tag{30}$$

is the same in any reference frames. You can easily verify it using Eq. (22),

$$\begin{aligned}(cd\tau')^2 &= (cdt')^2 - (dx')^2 - (dy')^2 - (dz')^2 \\ &= (\gamma cdt - \gamma\beta dz)^2 - (dx)^2 - (dy)^2 - (\gamma dz - \gamma\beta cdt)^2 \\ &= (\gamma^2 - \gamma^2\beta^2)(cdt)^2 - (dx)^2 - (dy)^2 - (\gamma^2 - \gamma^2\beta^2)(dz)^2 = (cd\tau)^2. \end{aligned} \tag{31}$$

A quantity that does not change from one reference frame to another is called a "Lorentz invariant" or an "invariant" for short. In this case, $d\tau$, a "proper time," is an invariant. Its physical meaning is the time interval in the rest frame of an object.

6 Four-vector Notation

We have seen that time and space get "mixed up" under Lorentz transformations. It is therefore useful to consider time and space different components of a single object, a four-component spacetime vector. The notation is $dx^\mu = (cdt, dx, dy, dz)$ where the Greek index μ runs from 0 ($dx^0 = cdt$) to 3 ($dx^3 = dz$). It is important that the index is a superscript. This is called four-vector notation.

Lorentz transformation is given by the matrix Eq. (23) acting on the four-vector dx^μ written as a column vector. Instead of using matrices and column vectors, the following notation is also used often,

$$(dx')^\mu = A^\mu{}_\nu dx^\nu. \tag{32}$$

In this notation, $A^\mu{}_\nu$ corresponds to the matrix A whose (μ, ν) component is $A^\mu{}_\nu$. A repeated index (ν in this case) is always summed over from 0 to 3. This is often called "Einstein's convention."[6] Note that the sum is over a lower index and an upper index. This is also an important aspect of the convention because of the reason we will see below. Any four-vector that transforms the same way as the spacetime four-vector is said to be "contravariant." As we will see in the next section, energy and momentum of an object are combined into a contravariant four-vector.

The proper time is an invariant. It is useful to introduce a "covariant" vector that has a lower index,

$$dx_\mu = (cdt, -dx, -dy, -dz). \tag{33}$$

Using this notation, the proper time can be written as

$$(cd\tau)^2 = dx_\mu dx^\mu. \tag{34}$$

The point is that if you sum over a lower and an upper index, the indices "cancel", namely two indices combine to an invariant. In the case of Lorentz transformation Eq. (32), the l.h.s. of the equation has only one index, while the r.h.s. three indices. The point is that the lower index in $A^\mu{}_\nu$ and the upper index in dx^ν are summed over and they "cancel." Effectively the r.h.s. also has only one index leftover and both sides of the equation are allowed to be equated.

The way a contravariant and a covarint vector are related to each other is given by a "metric tensor" $g_{\mu\nu}$,

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{35}$$

so that $dx_\mu = g_{\mu\nu} dx^\nu$. The inverse of the metric tensor has also the same form

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{36}$$

[6]Somebody told me that it wasn't Einstein at all who invented this convention. I'm not a historian to refute or verify this claim. Any volunteer?

so that $dx^\mu = g^{\mu\nu} dx_\nu$. The inverse relation can be written as $g_{\mu\nu} g^{\nu\lambda} = \delta_\mu^\lambda$ (Kronecker's delta which is an identity matrix). Spacetime with the metric of this form is called "Minkowsky space" as opposed to "Riemannian space" whose metric is positive definite.

7 Energy and Momentum Four-Vector

The action of a system must be Lorentz invaraint. The only invariant for a point particle is given by the proper time $d\tau$. Then the natural candidate for the action of a point particle is

$$S = -\int mc^2 d\tau. \tag{37}$$

You will see the reason for the factor $-mc^2$ shortly below.

Because of its definition Eq. (30), the action can be written explicitly as

$$S = -\int mc\sqrt{(cdt)^2 - (d\vec{x})^2} = -\int mc^2 \sqrt{1 - \frac{1}{c^2}\left(\frac{d\vec{x}}{dt}\right)^2} dt. \tag{38}$$

The Lagrangian is given by the integrand

$$L = -mc^2 \sqrt{1 - \frac{1}{c^2}\dot{\vec{x}}^2}. \tag{39}$$

In the non-relativistic limit, we can Taylor-expand the Lagrangian in $\dot{\vec{x}}/c$ to the second order and find

$$L = -mc^2 \left(1 - \frac{1}{2}\frac{1}{c^2}\dot{\vec{x}}^2\right) = -mc^2 + \frac{1}{2}m\dot{\vec{x}}^2. \tag{40}$$

This is indeed the Lagrangian of a non-relativistic point particle up to a constant term. The overall coefficient $-mc^2$ in the action was chosen to reproduce it.

Remember that the momentum is defined by

$$\vec{p} = \frac{\partial L}{\partial \dot{\vec{x}}}, \tag{41}$$

which gives $m\dot{\vec{x}}$ in the non-relativistc case. In our relativistic case,

$$\vec{p} = \frac{\partial}{\partial \dot{\vec{x}}} mc^2 \sqrt{1 - \frac{1}{c^2}\dot{\vec{x}}^2} = m\frac{\dot{\vec{x}}}{\sqrt{1 - \dot{\vec{x}}^2/c^2}} = mc\gamma\vec{\beta}. \tag{42}$$

It is useful again to check the non-relativistic limit $v \ll c$. By Taylor-expanding the expression for the momentum in $v/c = |\dot{\vec{x}}|/c$, the momentum is indeed $m\vec{v}$ up to corrections suppressed by $\vec{v}^2/c^2$.

The energy is given by the Hamiltonian $E = \vec{p} \cdot \dot{\vec{x}} - L$. In the non-relativistic case, it is

$$E = \vec{p} \cdot \dot{\vec{x}} - \frac{1}{2} m \dot{\vec{x}}^2 = \frac{1}{2} m \dot{\vec{x}}^2 = \frac{\vec{p}^2}{2m}. \tag{43}$$

On the other hand in the relativistic case,

$$E = \vec{p} \cdot \dot{\vec{x}} + mc^2 \sqrt{1 - \frac{1}{c^2} \dot{\vec{x}}^2} = \frac{mc^2}{\sqrt{1 - \dot{\vec{x}}^2/c^2}} = mc^2 \gamma. \tag{44}$$

Again by taking the non-relativistic limit, you find $E = mc^2 + \frac{1}{2} m \dot{\vec{x}}^2$. The first term is the rest energy of a particle $E = mc^2$.

By comparing the expressions of the energy and the momentum, it is easy to see that

$$\left(\frac{E}{c}\right)^2 - \vec{p}^2 = m^2 c^2. \tag{45}$$

No matter how fast the particle is moving, this combination is always the same. In other words, it is an invariant. This observation suggests that we can define a (contravariant) vector

$$p^\mu = (\frac{E}{c}, p_x, p_y, p_z). \tag{46}$$

The above relationship can then be rewritten as

$$p_\mu p^\mu = m^2 c^2. \tag{47}$$

To see that this interpretation indeed makes sense, let us look at its expression in the rest frame of the particle

$$p^\mu = (mc, 0, 0, 0). \tag{48}$$

Then going to the frame moving with velocity $-v$ along the z direction, the Lorentz transformation gives

$$(p')^\mu = (mc\gamma, 0, 0, mc\gamma\beta). \tag{49}$$

This precisely agrees E/c and $\vec{p}$ of the particle with what was obtained from the Lagrangian.

In relativistic kinematics seen in particle physics, the velocity is often very close to the speed of light, and it is not very useful to talk about it. Moreover, the velocity itself is not a conserved quantity under collision processes, while the energy and momentum are. Therefore, we almost exclusively talk about energy and momentum, very rarely about the velocity. Especially when we discuss massless particles (photon) or particles with tiny masses (electron, neutrinos), the velocity is always (approximately) the speed of light, yet energy can be any value. The formulae useful to remember are

$$E = \sqrt{c^2 \vec{p}^2 + m^2 c^4}, \qquad |\vec{p}| = \frac{1}{c}\sqrt{E^2 - m^2 c^4}. \tag{50}$$

If you need the velocity, you can get it as

$$v = c^2 \frac{|\vec{p}|}{E}. \tag{51}$$

It is amusing that the plane wave solution to the Schrödinger equation $e^{i\vec{p}\cdot\vec{x}/\hbar} e^{-iEt/\hbar}$ can be written as $e^{-ip_\mu x^\mu/\hbar}$. Secretly, the Schrödinger equation gave us a Lorentz-invariant expression for the wave function.[7]

8 Doppler Shift

As an application of Lorentz transformation and four-vectors, let us discuss Doppler shift of light emitted from moving bodies, for example far-away stars that are moving away from us because of expansion of Universe.[8]

The plane wave solution to Maxwell equation has time and space dependence $e^{-i\omega t + i\vec{k}\cdot\vec{x}}$. The exponent suggests that we can define contravariant wave four-vector $k^\mu = (\omega/c, \vec{k})$. The plane wave solution then has manifestly Lorentz-invariant form $e^{-ik_\mu x^\mu}$.[9] This is enough information for us to figure out the Doppler shift of light.

[7] Of course, the non-relativistic Schrödinger equation does not give the relativistic relation between the energy and the momentum.

[8] To correctly calculate the redshift in expanding Universe, however, general relativistic effects must also be taken into account.

[9] If you compare it to the quantum mechanical plane wave solution in the previous section, the relationship is obvious: $k_\mu = p_\mu/\hbar$.

The dispersion relation of light is $\omega = c|\vec{k}|$. For light propagating along the z-direction, therefore, the wave four-vector is simply $k^\mu = (\omega/c, \vec{k}) = (\omega/c)(1, 0, 0, 1)$. Suppose ω is the frequency of light in the rest frame of the emitter. If the light emitter is moving away from us with velocity v, all we need to do is to Lorentz-transform this four-vector. Using by-now familiar matrix Eq. (22), we find $(k')^\mu = (\omega/c)(\gamma - \gamma\beta, 0, 0, \gamma - \gamma\beta)$. In other words, the new (angular) frequency is related to the one in the rest frame of the light emitter by

$$\omega' = \omega(\gamma - \gamma\beta) = \omega \frac{1 - v/c}{\sqrt{1 - v^2/c^2}} = \omega \sqrt{\frac{1 - v/c}{1 + v/c}}. \tag{52}$$

This must be a familiar formula from Physics 7C.

Because light emitted from atoms has definite spectrum that can be measured in the laboratory, and spectroscopy can be done very accurately, measuring redshift of distant stars is a very powerful tool in astronomy.

9 Natural Unit

In the world of particle and nuclear physics where both relativity and quantum mechanics play crucial roles, it is cumbersome to keep writing the speed of light c and the reduced Planck constant $\hbar$. It is customary to set $c = \hbar = 1$. Why is this allowed?

Remember in the cgs system (also true in the MKS sytem), the fundamental units have dimensions of length (L), mass (M), and time (T). Other units are derived from these three. For example, ampère is defined by the force between two currents and it does not have to be a new unit. Indeed, in esu unit in the cgs system, electric charge and current are defined in terms of cm, g, and s only.

Given the three fundamental dimensions, we have the freedom of relating them among each other. For example, the length can be measured in the unit of time using the speed of light c which has dimension LT^{-1}. In daily life, it is extremely inconvenient to talk about "length of 33ps" if you mean just "a centimeter." (Note ps is pico-second, 10^{-12} sec.) But in the world of particle physics, it is not inconvenient at all. In fact, a typical size of a particle detector 1–10m translates to nanoseconds, which immediately put challenging demands on electronics. This way, L and T are now equivalent.

Similarly, the Planck constant has dimension L^2MT^{-1}. It can be used to relate the dimension of mass to other two. Most physicists use electronvolt eV as the unit for energy. Using the constant $\hbar c = 197$MeVfm (MeV is 10^6eV, fm is femto-meter 10^{-15}m), we can relate the length and the energy. Remember this constant! Any particle physicist remembers this number.

For example, we often refer to the mass of electron as $m_e = 0.511$ MeV$/c^2$. For short, we don't quote c^2. We just say the mass as 0.511 MeV. As long as it is understood that we always use c and $\hbar$ to relate different dimensions, there is no source of confusion. When the value is quoted in MeV, in order to get the dimension of mass, all you need to figure out is that you need a factor of $1/c^2$ (remember $E = mc^2$!). If you want, you can then work out the mass in SI unit, by using conversion constans $eV = 1.60\times10^{-19}$ J and $c = 3.00\times10^8$ m/s as $m_e = (0.511\times10^6\text{eV})(1.60\times10^{-19}\text{J/eV})/(3.00\times10^8\text{m/s})^2 = 9.08\times10^{-31}$kg. (Actually, it is 9.11×10^{-31}kg. We made this error because we kept only three significant digits.)

If the electron is moving with energy 1 MeV, we work out the momentum by

$$p = \sqrt{E^2 - m^2} = 0.860\text{MeV}. \tag{53}$$

What we mean is that the momentum is 0.860 MeV$/c$, but we don't write c. Only when you want to convert the momentum to the SI unit, you need to remember that there must be a factor of $1/c$. Otherwise, you can consistently drop c everywhere. This way, we quote values for energy, momentum, mass all in eV unit.

If you want the wave length λ, we of course use de Broglie's formula $\lambda = h/p = 2\pi\hbar/p$. But if you use natural unit, we quote p in eV, which is the same thing as writing it as $\lambda = 2\pi\hbar c/cp$. Then the only constant you need to remember is $\hbar c$. We immediately obtain $\lambda = 1440$ fm. This is the "resolution" of an experiment where 1 MeV electron is scattered.

There is another fundamental constant which can be used to eliminate any units: Newton's constant G_N. It has the dimension $L^3M^{-1}T^{-2}$. It can be used to construct a unit for energy, $E^2_{\text{Planck}} = \hbar c^3/G_N = (1.22\times10^{19}\text{GeV})^2$ called Planck energy. Correspondingly, the Planck length is $\ell_{\text{Planck}} = \hbar c/E_{\text{Planck}} = 1.61\times10^{-33}$ cm, an extremely short distance. Then any physical quantity can be expressed in terms of pure numbers with no dimensions. However, this energy is so enormous that it is not used as a unit even in particle physics. On the other hand, this energy (length) scale signals where quantum effects of gravity need to be considered. In fact, the candidate theory of quantum gravity, string theory, is often discussed using this unit.[10]

Note that it is not only particle and nuclear physicists who choose the unit system for convenience. Atomic physicists also use the same freedom to fix all three dimensions, by setting $e = m_e = \hbar = 1$. It is called atomic unit.

[10]Actually, what is used as the unit in string theory is not precisely the Planck unit, but a combination of string coupling constant and $E_{\text{Planck}}/\sqrt{8\pi}$ is set to unity.