HW #5

Harmonic Oscillator

(a)

We rewrite the Hamiltonian $H = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 x^2$ using $a = \sqrt{\frac{m\omega}{2\hbar}} (x + i \frac{p}{m\omega})$, $a^\dagger = \sqrt{\frac{m\omega}{2\hbar}} (x - i \frac{p}{m\omega})$. We first calculate

$$a^\dagger a = \frac{m\omega}{2\hbar} (x - i \frac{p}{m\omega})(x + i \frac{p}{m\omega}) = \frac{m\omega}{2\hbar} \left(x^2 - \frac{i}{m\omega}[p, x] + \frac{p^2}{m^2 \omega^2}\right) = \frac{m\omega}{2\hbar} \left(x^2 - \frac{\hbar}{m\omega} + \frac{p^2}{m^2 \omega^2}\right).$$

Therefore,

$$\hbar \omega a^\dagger a = \frac{1}{2} m \omega^2 x^2 - \frac{1}{2} \hbar \omega + \frac{p^2}{2m},$$

and hence $H = \hbar \omega (a^\dagger a + \frac{1}{2})$.

(b)

The ground state condition $a | 0 \rangle = 0$ can be written in the position representation as

$$\langle x | a | 0 \rangle = \left\langle x \middle| \sqrt{\frac{m\omega}{2\hbar}} (x + i \frac{p}{m\omega}) \middle| 0 \right\rangle = \sqrt{\frac{m\omega}{2\hbar}} (x + i \frac{1}{m\omega} \frac{\hbar}{i} \frac{d}{dx}) \langle x | 0 \rangle = 0,$$

and hence

$$(x + \frac{\hbar}{m\omega} \frac{d}{dx}) \psi_0(x) = 0.$$

This equation can be solved easily and we find

$$\psi_0(x) = N e^{-m\omega x^2 / 2\hbar}.$$

To normalize the wave function, we compute

$$\int_{-\infty}^{\infty} (e^{-m\omega x^2/2\hbar})^2 dx = \sqrt{\frac{\pi\hbar}{m\omega}}.$$

Therefore, the correctly normalized ground state wave function is

$$\psi_0(x) = (\frac{m\omega}{\pi\hbar})^{1/4} e^{-m\omega x^2/2\hbar}.$$

The shape of the wave function is

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ψ0[x_] := (m ω / (π ℏ))^(1/4) E^(-m ω x^2 / 2 ℏ)
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Plot[ψ₀[x] /. {m → 1, ω → 1, ℏ → 1}, {x, -3, 3}]
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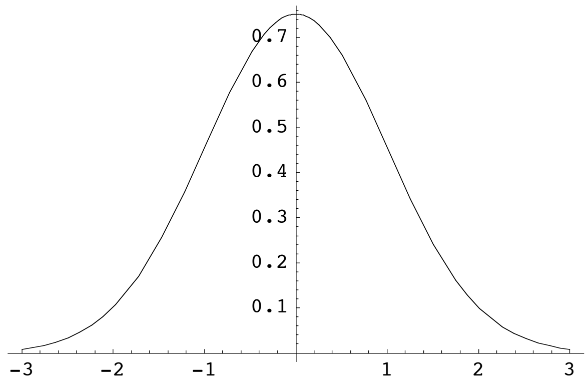

- Graphics -

(c)

The first excited state is given by $|1\rangle = a^\dagger |0\rangle$, and its position representation by

$$\langle x|1\rangle = \langle x|a^\dagger|0\rangle = \sqrt{\frac{m\omega}{2\hbar}}\,(x - i\,\frac{1}{m\omega}\,\frac{\hbar}{i}\,\frac{d}{dx})\,\langle x|0\rangle$$

$$= \sqrt{\frac{m\omega}{2\hbar}}\,(x+x)\,(\frac{m\omega}{\pi\hbar})^{1/4}\,e^{-m\omega x^2/(2\hbar)} = (\frac{m\omega}{\pi\hbar})^{1/4}\,\sqrt{\frac{m\omega}{2\hbar}}\,2\,x\,e^{-m\omega x^2/(2\hbar)}$$

Its shape is

$$\psi_1[x\_] := \left(\frac{m\,\omega}{\pi\,\hbar}\right)^{1/4}\sqrt{\frac{m\,\omega}{2\,\hbar}}\;2\,x\,E^{-m\,\omega\,x^2/(2\,\hbar)}$$

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Plot[ψ₁[x] /. {m → 1, ω → 1, ℏ → 1}, {x, -3, 3}]
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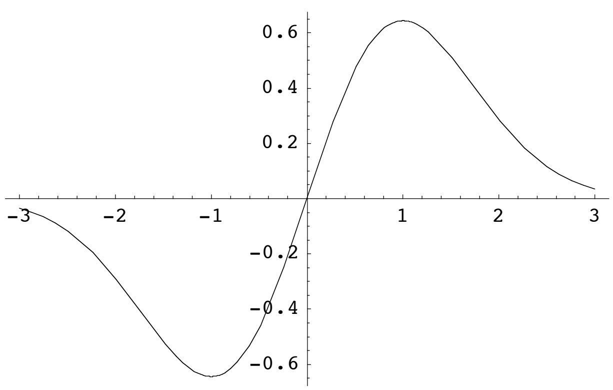

- Graphics -

Check that it is properly normalized:

$$\text{Integrate}\left[\psi_1[x]^2,\ \{x, -\infty, \infty\},\ \text{Assumptions} \to \text{Re}\left[\frac{m\,\omega}{\hbar}\right] > 0\right]$$

1

The second excited state is given by $\sqrt{2}\,\left|2\right\rangle = a^\dagger\,|1\rangle$, and its position representation by

$$\langle x\,|\,2\rangle = \frac{1}{\sqrt{2}}\,\langle x\,|\,a^\dagger\,|\,0\rangle = \frac{1}{\sqrt{2}}\,\sqrt{\frac{m\,\omega}{2\,\hbar}}\,\left(x - i\,\frac{1}{m\,\omega}\,\frac{\hbar}{i}\,\frac{d}{d\,x}\right)\langle x\,|\,1\rangle$$

Its shape is

$$\texttt{Simplify}\left[\frac{1}{\sqrt{2}}\,\sqrt{\frac{m\,\omega}{2\,\hbar}}\,\left(x\,\psi_1[x] - \frac{\hbar}{m\,\omega}\,D[\psi_1[x],\,x]\right)\right]$$

$$\frac{e^{-\frac{m\,x^2\,\omega}{2\,\hbar}}\,(2\,m\,x^2\,\omega - \hbar)\,\left(\frac{m\,\omega}{\hbar}\right)^{1/4}}{\sqrt{2}\,\pi^{1/4}\,\hbar}$$

$$\psi_2[x\_] := \frac{e^{-\frac{m\,x^2\,\omega}{2\,\hbar}}\,(2\,m\,x^2\,\omega - \hbar)\,\left(\frac{m\,\omega}{\hbar}\right)^{1/4}}{\sqrt{2}\,\pi^{1/4}\,\hbar}$$

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Plot[ψ2[x] /. {m → 1, ω → 1, ℏ → 1}, {x, -3, 3}]
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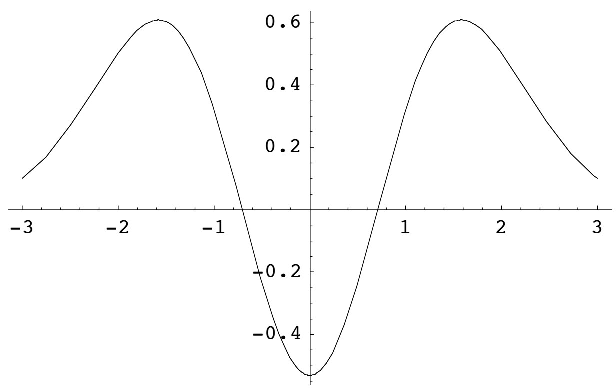


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Check that it is properly normalized:

$$\texttt{Integrate}\left[\psi_2[x]^2,\,\{x,\,-\infty,\,\infty\},\,\texttt{Assumptions} \to \texttt{Re}\left[\frac{m\,\omega}{\hbar}\right] > 0\right]$$

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1
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(d)

From the definitions of the annihilation and creation operators, we can solve for x,

$$x = \sqrt{\frac{\hbar}{2\,m\,\omega}}\,(a + a^\dagger).$$

Starting with the expectation values,

$$\langle x\rangle = \langle n\,|\,x\,|\,n\rangle = \sqrt{\frac{\hbar}{2\,m\,\omega}}\,\langle n\,|\,a + a^\dagger\,|\,n\rangle$$

$$= \sqrt{\frac{\hbar}{2\,m\,\omega}}\,\left\langle n\,\middle|\,\left(\sqrt{n}\,\left|n-1\right\rangle + \sqrt{n+1}\,\left|n+1\right\rangle\right)\right. = 0$$

because of the orthonormality of the Hamiltonian eigenstates $\langle n\,|\,m\rangle = \delta_{n,m}$.

Moving on to the variance,

$$\langle x^2\rangle = \frac{\hbar}{2\,m\,\omega}\,\left\langle n\,\middle|\,(a + a^\dagger)^2\,\middle|\,n\right\rangle = \frac{\hbar}{2\,m\,\omega}\,\langle n\,|\,a^\dagger\,a + a\,a^\dagger\,|\,n\rangle,$$

$$= \frac{\hbar}{2\,m\,\omega}\,\langle n\,|\,2\,N + [a,\,a^\dagger]\,|\,n\rangle = \frac{\hbar}{2\,m\,\omega}\,(2\,n + 1).$$

(e)

From the definitions of the annihilation and creation operators, we can solve for p,

$$p = -i\sqrt{\frac{\hbar m \omega}{2}}\,(a - a^{\dagger}).$$

Together with the expression for x from the previous problem, you can easily verify $[x, p] = i\hbar$. Starting with the expectation values,

$$\langle p \rangle = \langle n \,|\, p \,|\, n \rangle = -i\sqrt{\frac{\hbar m \omega}{2}}\,\langle n \,|\, a - a^{\dagger} \,|\, n \rangle = 0.$$

Moving on to the variance,

$$\langle p^2 \rangle = -\frac{\hbar m \omega}{2}\left\langle n \,\middle|\, (a - a^{\dagger})^2 \,\middle|\, n \right\rangle = \frac{\hbar m \omega}{2}\langle n \,|\, a^{\dagger} a + a a^{\dagger} \,|\, n \rangle,$$

$$= \frac{\hbar m \omega}{2}\langle n \,|\, 2N + [a, a^{\dagger}] \,|\, n \rangle = \frac{\hbar m \omega}{2}(2n + 1).$$

Therefore,

$$(\Delta x)(\Delta p) = \frac{\hbar}{2}(2n + 1).$$

The ground state $n = 0$ is a minimum uncertainty state, while the excited states have larger uncertainties.

(f)

There are many ways to show this.

First of all, using the relation $a^\dagger \left| n \right\rangle = \sqrt{n+1} \left| n+1 \right\rangle$, we can show that

$$(a^\dagger)^n \left| 0 \right\rangle = \sqrt{n!} \left| n \right\rangle$$

by recursion. It obviously holds for $n = 1$. If it holds for n, the $n+1$-th one is

$(a^\dagger)^{n+1} \left| 0 \right\rangle = a^\dagger (a^\dagger)^n \left| 0 \right\rangle = a^\dagger \sqrt{n!} \left| n \right\rangle = \sqrt{n!} \sqrt{n+1} \left| n+1 \right\rangle = \sqrt{(n+1)!} \left| n+1 \right\rangle$, and it holds again. Therefore, it holds for all n.

Using the definition $\left| f \right\rangle = \sum_{n=0}^{\infty} \frac{f^n}{n!} (a^\dagger)^n \left| 0 \right\rangle$, we can write it as $\left| f \right\rangle = \sum_{n=0}^{\infty} \frac{f^n}{\sqrt{n!}} \left| n \right\rangle$. Then

$$a \left| f \right\rangle = a \sum_{n=0}^{\infty} \frac{f^n}{\sqrt{n!}} \left| n \right\rangle = \sum_{n=1}^{\infty} \frac{f^n}{\sqrt{n!}} \sqrt{n} \left| n-1 \right\rangle,$$

where the summation is now taken only from $i = 1$ because $i = 0$ term vanishes by $a \left| 0 \right\rangle = 0$. Continuing on,

$$a \left| f \right\rangle = \sum_{n=1}^{\infty} \frac{f^n}{\sqrt{(n-1)!}} \left| n-1 \right\rangle = \sum_{m=0}^{\infty} \frac{f^{m+1}}{\sqrt{m!}} \left| m \right\rangle,$$

where the dummy variable was chgned to $m = n - 1$. Pulling one factor of f ouf of the sum,

$$a \left| f \right\rangle = f \sum_{m=0}^{\infty} \frac{f^m}{\sqrt{m!}} \left| m \right\rangle = f \left| f \right\rangle.$$

Another way to show the same result is by first showing the relation

$$[a, (a^\dagger)^n] = n(a^\dagger)^{n-1}.$$

It obviously holds for $n = 1$. If it holds for n, the $n+1$-th one is

$$\left[a, (a^\dagger)^{n+1}\right] = [a, a^\dagger (a^\dagger)^n] = [a, a^\dagger] (a^\dagger)^n + a^\dagger [a, (a^\dagger)^n]$$

$$= (a^\dagger)^n + a^\dagger\, n(a^\dagger)^n = (n+1)(a^\dagger)^{n+1}$$

and hence it holds as well. Therefore, it holds for any n.

Starting with the definition $\left| f \right\rangle = \sum_{n=0}^{\infty} \frac{f^n}{n!} (a^\dagger)^n \left| 0 \right\rangle$,

$$a \left| f \right\rangle = a \sum_{n=0}^{\infty} \frac{f^n}{n!} (a^\dagger)^n \left| 0 \right\rangle = \sum_{n=0}^{\infty} \frac{f^n}{n!} [a, (a^\dagger)^n] \left| 0 \right\rangle,$$

Here, we used the fact $a \left| 0 \right\rangle = 0$. Using the relation shown above,

$$a \left| f \right\rangle = \sum_{n=1}^{\infty} \frac{f^n}{n!}\, n\, (a^\dagger)^{n-1} \left| 0 \right\rangle = \sum_{n=1}^{\infty} \frac{f^n}{(n-1)!} (a^\dagger)^{n-1} \left| 0 \right\rangle,$$

where the summation is now taken only from $i = 1$ because $i = 0$ term vanishes by $[a, 1] = 0$. Changing the dummy variable to $m = n - 1$,

$$a \left| f \right\rangle = \sum_{m=0}^{\infty} \frac{f^{m+1}}{m!} (a^\dagger)^m \left| 0 \right\rangle = f \sum_{m=0}^{\infty} \frac{f^m}{m!} (a^\dagger)^m \left| 0 \right\rangle = f \left| f \right\rangle,$$

(g)

Using the result from (d),

$$\langle f \mid x \mid f \rangle = \left\langle f \,\middle|\, \sqrt{\frac{\hbar}{2 m \omega}} \, (a + a^\dagger) \,\middle|\, f \right\rangle = \sqrt{\frac{\hbar}{2 m \omega}} \, (f + f^*) = \sqrt{\frac{\hbar}{2 m \omega}} \, 2 \operatorname{Re}(f).$$

Here, we used the fact $\langle f \mid a^\dagger = \langle f \mid f^*$, obtained by taking the hermitian conjugate of $a \mid f \rangle = f \mid f \rangle$. Similarly, using the result from (e),

$$\langle f \mid p \mid f \rangle = \left\langle f \,\middle|\, -i \sqrt{\frac{\hbar m \omega}{2}} \, (a - a^\dagger) \,\middle|\, f \right\rangle = -i \sqrt{\frac{\hbar m \omega}{2}} \, (f - f^*) = \sqrt{\frac{\hbar m \omega}{2}} \, 2 \operatorname{Im}(f).$$

Now on the variance,

$$\begin{aligned} \langle f \mid x^2 \mid f \rangle &= \left\langle f \,\middle|\, \frac{\hbar}{2 m \omega} \, (a + a^\dagger)^2 \,\middle|\, f \right\rangle = \frac{\hbar}{2 m \omega} \left\langle f \,\middle|\, a^2 + a\, a^\dagger + a^\dagger \, a + (a^\dagger)^2 \,\middle|\, f \right\rangle \\ &= \frac{\hbar}{2 m \omega} \left\langle f \,\middle|\, a^2 + [a,\ a^\dagger] + 2\, a^\dagger \, a + (a^\dagger)^2 \,\middle|\, f \right\rangle = \frac{\hbar}{2 m \omega} \, (f^2 + 1 + 2\, f^* \, f + (f^*)^2) \\ &= \frac{\hbar}{2 m \omega} \, ((f + f^*)^2 + 1) \end{aligned}$$

and hence

$$(\Delta\, x)^2 = \frac{\hbar}{2 m \omega}.$$

Similarly,

$$\begin{aligned} \langle f \mid p^2 \mid f \rangle &= \left\langle f \,\middle|\, -\frac{\hbar m \omega}{2} \, (a - a^\dagger)^2 \,\middle|\, f \right\rangle = \frac{\hbar m \omega}{2} \left\langle f \,\middle|\, -a^2 + a\, a^\dagger + a^\dagger \, a - (a^\dagger)^2 \,\middle|\, f \right\rangle \\ &= \frac{\hbar m \omega}{2} \left\langle f \,\middle|\, -a^2 + [a,\ a^\dagger] + 2\, a^\dagger \, a - (a^\dagger)^2 \,\middle|\, f \right\rangle = \frac{\hbar m \omega}{2} \, (-f^2 + 1 + 2\, f^* \, f - (f^*)^2) \\ &= \frac{\hbar m \omega}{2} \, (-(f - f^*)^2 + 1) \end{aligned}$$

and hence

$$(\Delta\, p)^2 = \frac{\hbar m \omega}{2}.$$

Therefore,

$$(\Delta\, x)\, (\Delta\, p) = \frac{\hbar}{2}$$

and hence the coherent state is a minimum uncertainty state for any f.

(h)

The Schrödinger equation gives $\mid n,\, t \rangle = e^{-i H t/\hbar} \mid n \rangle = e^{-i \hbar \omega (n+1/2) t/\hbar} \mid n \rangle = e^{-i \omega (n+1/2) t} \mid n \rangle$. Using what we showed above, $\mid f \rangle = \sum_{n=0}^{\infty} \frac{f^n}{\sqrt{n!}} \left| n \right\rangle$, its time evolution is

$$\begin{aligned} \mid f,\, t \rangle &= e^{-i H t/\hbar} \sum_{n=0}^{\infty} \frac{f^n}{\sqrt{n!}} \left| n \right\rangle = \sum_{n=0}^{\infty} \frac{f^n}{\sqrt{n!}} \left| n \right\rangle e^{-i \omega (n+1/2) t} \\ &= \sum_{n=0}^{\infty} \frac{(f\, e^{-i \omega t})^n}{\sqrt{n!}} \left| n \right\rangle e^{-i \omega t/2} = \left| f\, e^{-i \omega t} \right\rangle e^{-i \omega t/2} \end{aligned}$$

where the coherent state in the last expression has the eigenvalue $a \mid f\, e^{-i \omega t} \rangle = f\, e^{-i \omega t} \mid f\, e^{-i \omega t} \rangle$. The exectation value of the position operator is

$$\begin{aligned} \langle f,\, t \mid x \mid f,\, t \rangle &= e^{i \omega t/2} \langle f\, e^{-i \omega t} \mid x \mid f\, e^{-i \omega t} \rangle\, e^{-i \omega t/2} = \left\langle f\, e^{-i \omega t} \,\middle|\, \sqrt{\frac{\hbar}{2 m \omega}} \, (a + a^\dagger) \,\middle|\, f\, e^{-i \omega t} \right\rangle \\ &= \sqrt{\frac{\hbar}{2 m \omega}} \, (f\, e^{-i \omega t} + f^* \, e^{i \omega t}) \langle f\, e^{-i \omega t} \mid f\, e^{-i \omega t} \rangle = \sqrt{\frac{\hbar}{2 m \omega}} \, 2\, (\operatorname{Re}(f) \cos \omega\, t + \operatorname{Im}(f) \sin \omega\, t) \langle f,\, t \mid f,\, t \rangle \end{aligned}$$

Therefore, the expectation value is

$$\langle x \rangle\, (t) = \frac{\langle f,t|x|f,t \rangle}{\langle f,t|f,t \rangle} = \sqrt{\frac{\hbar}{2 m \omega}} \, 2\, (\operatorname{Re}(f) \cos \omega\, t + \operatorname{Im}(f) \sin \omega\, t),$$

and shows the oscillatory behavior just like the classical solution.

(i)

The Heisenberg equation of motion is

$$i\hbar \frac{d}{dt} x = [x, H] = \left[x, \frac{p^2}{2m}\right] = i\hbar \frac{p}{m},$$

$$i\hbar \frac{d}{dt} p = [p, H] = [p, \tfrac{1}{2} m\omega^2 x^2] = -i\hbar m\omega^2 x.$$

There are many ways to solve these coupled equations. One way is to use the exponential of a matrix as Daniel explained in the section. Write the equations in the matrix form,

$$\frac{d}{dt}\begin{pmatrix} m\omega x \\ p \end{pmatrix} = \begin{pmatrix} \omega p \\ -m\omega^2 x \end{pmatrix} = \omega\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} m\omega x \\ p \end{pmatrix}.$$

Therefore,

$$\begin{pmatrix} m\omega x \\ p \end{pmatrix}(t) = \exp\left(\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} i\omega t\right)\begin{pmatrix} m\omega x \\ p \end{pmatrix}(0) = \exp(\sigma_2\, i\omega t)\begin{pmatrix} m\omega x \\ p \end{pmatrix}(0).$$

Here, σ_2 is one of the Pauli matrices. The exponential factor can be worked out using its Taylor expansion,

$$\exp(\sigma_2\, i\omega t) = \sum_{n=0}^{\infty} \frac{1}{n!} \sigma_2{}^n (i\omega t)^n.$$

It is easy to check that $\sigma_2{}^2 = 1$, and hence $\sigma_2{}^{\text{even}} = 1$, $\sigma_2{}^{\text{odd}} = \sigma_2$. Therefore,

$$\exp(\sigma_2\, i\omega t) = \sum_{n=0,\,\text{even}}^{\infty} \frac{1}{n!} 1\,(i\omega t)^n + \sum_{n=0,\,\text{odd}}^{\infty} \frac{1}{n!} \sigma_2 (i\omega t)^n$$

$$= 1\cos\omega t + i\sigma_2 \sin\omega t = \begin{pmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{pmatrix}.$$

We find the solution

$$\begin{pmatrix} m\omega x \\ p \end{pmatrix}(t) = \begin{pmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{pmatrix}\begin{pmatrix} m\omega x \\ p \end{pmatrix}(0)$$

$$= \begin{pmatrix} m\omega x(0)\cos\omega t + p(0)\sin\omega t \\ -m\omega x(0)\sin\omega t + p(0)\cos\omega t \end{pmatrix}.$$

Using this solution, we calculate the expectation value,

$$\langle x\rangle(t) = \frac{\langle f|x(t)|f\rangle}{\langle f|f\rangle} = \frac{1}{\langle f|f\rangle}\langle f\,|\,x(0)\cos\omega t + \frac{p(0)}{m\omega}\sin\omega t\,|\,f\rangle$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\; 2\,\mathrm{Re}(f)\cos\omega t + \frac{1}{m\omega}\sqrt{\frac{\hbar m\omega}{2}}\; 2\,\mathrm{Im}(f)\sin\omega t$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\; 2\,(\mathrm{Re}(f)\cos\omega t + \mathrm{Im}(f)\sin\omega t),$$

which agrees with the calculation in the Schrödinger picture.