Final Exam (221A), due Dec 17, 4pm

1. The sodium D-lines are two separate emission lines for $3p \to 3s$ transitions. In a weak magnetic field, consider the Zeeman effect and discuss how the emission lines are split quantitatively.[20]

2. A positive ion with one outer-shell electron in a p-orbital is surrounded by four negative ions. Taking the position of the positive ion as the origin, the negative ions are at $(a, 0, 0)$, $(-a, 0, 0)$, $(0, a, 0)$, and $(0, -a, 0)$. Regard the negative ions as point particles of charge $e = -|e|$. Ignore the electron spin.

   (a) Work out the potential energy of the electron at (x, y, z) due to the negative ions assuming $|x|, |y|, |z| \ll a$ to the second order in x, y, and z.[10]

   (b) Calculate how the p-orbitals are split due to the above perturbation in terms of $\langle r^2 \rangle = \int_0^\infty drr^4 R^2(r)$. If there is any degeneracy, identify the symmetry responsible for the degeneracy.[15]

3. A tritium ^{3}H is a hydrogen-like atom with a nucleus made of one proton and two neutrons t. The nucleus undergoes the β-decay $t \to {}^3\text{He} + e^- + \bar{\nu}_e$. Using the sudden approximation and ignoring the recoil of the nucleus, calculate the probabilities to find the resulting He$^+$ ion in the $1s$, $2s$, and $2p$ states.[15]

4. One useful way to use the Dyson series is to identify the energy shifts due to a perturbation, even when it is time-dependent.

   (a) When V is time-independent, work out $\langle i|U_I(t)|i\rangle$ to the second order, and identify $\Delta^{(1)}$, $\Delta^{(2)}$, and the wave function renormalization Z_i in the expansion of

$$\begin{aligned} \langle i|U_I(t)|i\rangle &= Z_i e^{-i\Delta E t/\hbar} + \text{rapidly oscillating pieces} \\ &= Z_i + \frac{-i}{\hbar}(\Delta_i^{(1)} + \Delta_i^{(2)})t + \frac{1}{2!}\left(\frac{-i}{\hbar}\Delta_i^{(1)} t\right)^2 + O(V^3) \end{aligned} \quad (1)$$

   and show that they agree with the results from the time-independent perturbation theory, Eqs. (5.1.42), (5.1.44), and (5.1.48b) in Sakurai. Note that this identification is done in the $t \to \infty$ limit where rapidly oscillating terms are dropped.[10]

(b) Now consider a harmonic perturbation $V = V_0 \cos \omega t$. Work out the second-order energy shift.[10]

(c) Use this formula to calculate the polarizability, and show that it is given by[10]

$$\alpha(\omega) = -2e^2 \sum_{k \neq 0}^{\infty} \frac{|\langle k^{(0)}|z|1,0,0\rangle|^2 (E_0^{(0)} - E_k^{(0)})}{(E_0^{(0)} - E_k^{(0)})^2 - (\hbar\omega)^2}. \tag{2}$$

(d) Discuss if the index of refraction increases or decreases for visible light ($\hbar\omega < |E_i - E_k|$) of shorter wave length $\lambda = 2\pi c/\omega$, and predict the order of colors in a rainbow.[10]