HW #9 (221A), due Nov 10, 4pm

1. Consider a three-dimensional isotropic harmonic oscillator with Hamiltonian
$$H = \frac{\vec{p}^2}{2m} + \frac{1}{2} m\omega^2 \vec{x}^2 = \hbar\omega \left(\vec{a}^\dagger \cdot \vec{a} + \frac{3}{2} \right).$$
(This is the starting point of the shell model of nuclei.) Answer the following questions.

   (a) Clearly, the system is spherically symmetric, and hence there is a conserved angular momentum vector. Show that $\vec{L} = \vec{x} \times \vec{p}$ commutes with the Hamiltonian.

   (b) Rewrite $\vec{L}$ in terms of creation and annihilation operators.

   (c) Show that $|0\rangle$ belongs to the $l = 0$ representation. It is called 1S state.

   (d) Show that the operators $\mp(a_x^\dagger \pm i a_y^\dagger)$ and $a_z^\dagger$ form spherical tensor operators of $k = 1$.

   (e) Show that the $N = 1$ states, $|1, 1, \pm 1\rangle = \mp(a_x^\dagger \pm i a_y^\dagger)|0\rangle/\sqrt{2}$ and $|1, 1, 0\rangle = a_z^\dagger|0\rangle$, form the $l = 1$ representation. (Notation is $|N, l, m\rangle$.) It is called 1P state because it is the first P-state.

   (f) Calculate the expectation values of the quadrupole moment $Q = (3z^2 - r^2)$ for $N = l = 1$, $m = -1, 0, 1$ states, and verify the Wigner–Eckart theorem.

   (g) There are six possible states at $N = 2$ level. Construct states $|2, l, m\rangle$ with definite $l = 0, 2$ and m. They are called 2S (because it is the second S-state) and 1D (because it is the first D-state).

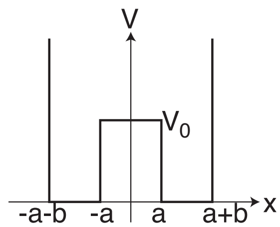
   (h) How many possible states are there at $N = 3, 4$ levels? What l representations do they fall into?

   (i) (optional) What about general N?

   (j) (optional) Verify that the operator $\Pi = e^{i\pi \vec{a}^\dagger \cdot \vec{a}}$ has the correct property as the parity operator by showing $\Pi \vec{x} \Pi^\dagger = -\vec{x}$, $\Pi \vec{p} \Pi^\dagger = -\vec{p}$.

   (k) Show that $\Pi = (-1)^N$.

   (l) Without calculating it explicitly, show that there is no dipole transition from 2P to 1P state.



2. Consider a particle of mass m in a symmetric (parity-invariant) rectangular double-well potential:

$$V = \begin{cases} \infty & \text{for } |x| > a+b; \\ 0 & \text{for } a < |x| < a+b; \\ V_0 > 0 & \text{for } |x| < a. \end{cases} \tag{1}$$

Assuming that V_0 is very high compared to the quantized energies of low-lying states, obtain an approximate expression for the energy splitting between the two lowest-lying states. (It must be exponentially small confirming that it is a tunneling effect.) Also plot their wave functions for appropriate choice of parameters.

3. (optional) Rotation spectra of diatomic molecules can be understood approximately by the Hamiltonian

$$H = \frac{\vec{J}^2}{2I}, \tag{2}$$

where $\vec{J}$ is the angular momentum in the body frame. (For a better fit to the observed spectra for high levels, one needs to consider "centrifugal corrections" that are higher orders in $\vec{J}$, which we ignore in this problem.) Look up the spectrum of ^{12}C ^{32}S for transitions between $J = 0$ to 11 states and verify that the spectrum approximately falls in to the expected one. Determine I and the interatomic distance between carbon and sulpher. Next, compare the spectrum for different isotopes ^{12}C ^{34}S and ^{12}C ^{36}S and explain the difference. The spectra can be obtained from `http://physics.nist.gov/PhysRefData/MolSpec/Diatomic/index.html` Focus on the lowest vibrational state (*i.e.*, column "Vib." with entry "0.")