221A Lecture Notes
Notes on Classica Mechanics II

1 Hamilton–Jacobi Equations

The use of action does not stop in obtaining Euler–Lagrange equation in classical mechanics. Instead of using the action to vary in order to obtain the equation of motion, we can regard the action as a function of the end point by *using* the solution to the equation of motion. This may sound like an odd thing to do, but turns out to be another useful formulation of classical mechanics.

1.1 Free Particle in One Dimension

Think about a free particle in one dimension for the moment. The Lagrangian is simply $L = \frac{m}{2}\dot{x}^2$. For the initial position x_i at time t_i, and the final position x_f at time t_f, the equation of motion can be easily solved and we find

$$x(t) = x_i + \frac{x_f - x_i}{t_f - t_i}(t - t_i). \tag{1}$$

If you insert this solution to the action, we find

$$S(x_f, t_f; x_i, t_i) = \frac{m}{2}\frac{(x_f - x_i)^2}{t_f - t_i}. \tag{2}$$

This expression is interesting, because it gives

$$\frac{\partial S}{\partial x_f} = m\frac{x_f - x_i}{t_f - t_i} = mv = p \tag{3}$$

$$\frac{\partial S}{\partial t_f} = -\frac{m}{2}\frac{(x_f - x_i)^2}{(t_f - t_i)^2} = -\frac{1}{2}mv^2 = -E. \tag{4}$$

Here, p is the momentum and E the energy for this solution.

If we had known this point before hand, we could have written the following equation (Hamilton–Jacobi equation),

$$E = H(p, q) = \frac{p^2}{2m}, \tag{5}$$

and hence

$$\frac{\partial S}{\partial t_f} + \frac{1}{2m}\left(\frac{\partial S}{\partial x_f}\right)^2 = 0. \qquad (6)$$

It is easy to see that the expression for the action obtained above is a solution to this equation.

1.2 Hamilton–Jacobi Equation

In general, we can regard the action a function of the final position q_i and time t, keeping the intial data fixed. Then we can show

$$\frac{\partial S}{\partial q_i} = p_i, \qquad \frac{\partial S}{\partial t} = -H. \qquad (7)$$

(Here, we already see the connection between the momentum and space-derivative, and the energy and the time-derivative, hinting at what we do in quantum mechanics.) Then one can write the Hamilton–Jacobi equation

$$\frac{\partial S}{\partial t} + H\left(\frac{\partial S}{\partial q}, q\right) = 0 \qquad (8)$$

using the Hamiltonian $H(p, q)$.

Here is how we see Eq. (7). First of all, when we change the end point of the motion $q_i(t_f)$ to $q_i(t_f)+\delta q_i$, the entire trajectory is changed to $q_i(t)+\delta q_i(t)$ with the boundary conditions $\delta q_i(t_i) = 0$, $\delta q_i(t_f) = \delta q_i$. Remember we evaluate the action along the trajectory that satisifes the equation of motion. The action changes by

$$\begin{aligned}
\delta S &= \int_{t_i}^{t+f} \left(\frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i\right) dt \\
&= \int_{t_i}^{t+f} \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i\right) dt \\
&= \left[\frac{\partial L}{\partial \dot{q}_i}\delta q_i\right]_{t_i}^{t_f} \\
&= \frac{\partial L}{\partial \dot{q}_i}(t_f)\delta q_i. \qquad (9)
\end{aligned}$$

In the second one we used the equation of motion. Therefore, we find

$$\frac{\partial S}{\partial q_i} = p_i(t_f). \qquad (10)$$

The variation with respect to t_f needs to be done carefully. When we fix the end point of the motion q_i but change the arrival time t_f to $t_f + \delta t$, we need to change $q_i(t_f)$ to $q_i(t_f) - \dot{q}_i \delta t$ so that it arrives at the same q_i at time $t_f + \delta t$. Therefore, there are two contributions to δS. One is just because of the change in the end point of the time integral $L(t_f)\delta t$, and the other due to the change in $q_i(t_f)$, and hence

$$\delta S = L(t_f)\delta t + \frac{\partial L}{\partial \dot{q}_i}(t_f)(-\dot{q}_i \delta t) = -H\delta t. \tag{11}$$

This proves Eq. (7).

Why do we formulate the classical mechanics this way? Well, it turns out that this is probably the easiest method to solve Kepler motion (or hydrogen atom at the classical level). The point is that solving differential equation in three-dimensional space is not easy. We of course know the answer to the Kepler problem, but if you have tried to work out the elliptic orbit yourself, you know it ain't easy! The Hamilton–Jacobi method makes the mechanics problem mechanical. If you can completely separate variables (you'll see below what I mean), the problem reduces to simple integrals. It helps a lot that there are no second-order derivatives in the equation.

Skip this if you are not familiar with general relativity. When you solve for particle trajectory in general relativity (in curved space-time), we would like the equation to be invariant under general coordinate transformations. The Hamilton–Jacobi equation is given by

$$g^{\mu\nu} \frac{\partial S}{\partial x^\mu} \frac{\partial S}{\partial x^\nu} = m^2 c^2, \tag{12}$$

and is indeed fully invarant under general cooredinate transformations. This is another useful application of Hamilton–Jacobi equation.

1.3 Harmonic Oscillator

Let us apply Hamilton–Jacobi method to a harmonic oscillator. Of course, a harmonic oscillator can be easily solved using the conventional equation of motion, but this excercise would be useful to understand the basic method.

From the Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 q^2, \tag{13}$$

the Hamilton–Jacobi equation is written down as

$$\frac{\partial S}{\partial t} + \frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 + \frac{1}{2} m\omega^2 q^2 = 0. \tag{14}$$

Because the equation does not explicit involve t, we can write

$$S(q,t) = \tilde{S}(q,E) - Et \tag{15}$$

to obtain

$$\frac{1}{2m}\left(\frac{\partial \tilde{S}}{\partial q}\right)^2 + \frac{1}{2}m\omega^2 q^2 = E. \tag{16}$$

$\tilde{S}$ can be obtained easily from this equation

$$\tilde{S} = \int \sqrt{2mE - m^2\omega^2 q^2}\, dq = \frac{E}{\omega}\left(\arcsin\frac{m\omega q}{\sqrt{2mE}} + \frac{m\omega q}{\sqrt{2mE}}\sqrt{1 - \left(\frac{m\omega q}{\sqrt{2mE}}\right)^2}\right). \tag{17}$$

Note that the change from S to $\tilde{S}$ can be viewed as a Legendre transform. We will use the inverse Legendre transform to find the time-dependence of the motion

$$t = \frac{\partial \tilde{S}}{\partial E} = \frac{1}{\omega}\arcsin\frac{m\omega q}{\sqrt{2mE}}. \tag{18}$$

This is indeed what we want:

$$q = \sqrt{\frac{2E}{m}}\sin\omega t. \tag{19}$$

The momentum is given by

$$p = \frac{\partial S}{\partial q} = \sqrt{2mE - m^2\omega^2 q^2} \tag{20}$$

as required.

1.4 Motion in a Central Potential

When a particle is moving in a central potential $V(r)$, a function only of the radius r, the Hamilton–Jacobi equation can be solved by using the spherical coordinates. The Lagrangian is

$$L = \frac{m}{2}\dot{\vec{x}}^2 - V(r). \tag{21}$$

Going to the spherical coordinates, it becomes

$$L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2) - V(r). \tag{22}$$

The canonical momenta are defined as

$$p_r = m\dot{r}, \qquad p_\theta = mr^2\dot{\theta}, \qquad p_\phi = mr^2 \sin^2\theta\dot{\phi}. \tag{23}$$

Following the definition, we find the Hamiltonian

$$H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{p_\phi^2}{2mr^2\sin^2\theta} + V(r). \tag{24}$$

Then the Hamilton–Jacobi equation is found to be

$$\frac{\partial S}{\partial t} + \frac{1}{2m}\left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{2mr^2}\left(\frac{\partial S}{\partial \theta}\right)^2 + \frac{1}{2mr^2\sin^2\theta}\left(\frac{\partial S}{\partial \phi}\right)^2 + V(r) = 0. \tag{25}$$

It still looks complicated, but it can be drastically simplified using the so-called separation of variables.

Separation of variables is done in the following simple manner:

$$S(t, r, \theta, \phi) = S_1(t) + S_2(r) + S_3(\theta) + S_4(\phi). \tag{26}$$

Then the Hamilton–Jacobi equation becomes

$$\frac{dS_1}{dt} + \frac{1}{2m}\left(\frac{dS_2}{dr}\right)^2 + \frac{1}{2mr^2}\left(\frac{dS_3}{d\theta}\right)^2 + \frac{1}{2mr^2\sin^2\theta}\left(\frac{dS_4}{d\phi}\right)^2 + V(r) = 0. \tag{27}$$

Because there are no explicit t- and ϕ-dependence in the equation, we conclude dS_1/dt, $dS_4/d\phi$ must be constant. We set

$$\frac{dS_1}{dt} = -E, \qquad \frac{dS_4}{d\phi} = L_z. \tag{28}$$

They indeed have the meaning of the energy and the z-component of the orbital angular momentum, as we will see later. Then

$$S_2(r) + S_3(\theta) = S(t, r, \theta, \phi) + Et - L_z\phi, \tag{29}$$

a Legendre transformation from the original action S. The Hamilton–Jacobi equation is now

$$\frac{1}{2m}\left(\frac{dS_2}{dr}\right)^2 + \frac{1}{2mr^2}\left(\frac{dS_3}{d\theta}\right)^2 + \frac{L_z^2}{2mr^2\sin^2\theta} + V(r) = E. \tag{30}$$

θ-dependence is only in the second and third terms in the left-hand side of the equation. Therefore, we must have the following combination constant,

$$\left(\frac{dS_3}{d\theta}\right)^2 + \frac{L_z^2}{\sin^2\theta} = L^2. \tag{31}$$

Again L has the meaning of the orbital angular momentum, as can be seen as follows. $S_3(\theta)$ is obtained by integrating

$$\frac{dS_3}{d\theta} = \sqrt{L^2 - \frac{L_z^2}{\sin^2\theta}}\ . \tag{32}$$

This equation makes it clear that $L^2 \geq L_z^2$ in order for a solution to exist. Because the change from the original action S to S_2+S_3 Eq. (29) is a Legendre transform, the inverse transform can be used to determine ϕ by a derivative with respect to L_z. Note that S_2 does not depend on L_z (see below, Eq. (36), and hence the only L_z dependence appears in $S_3(\theta)$. Therefore,

$$\begin{aligned}\phi &= -\frac{\partial S_3}{\partial L_z} = -\frac{\partial}{\partial L_z}\int\sqrt{L^2 - \frac{L_z^2}{\sin^2\theta}}\,d\theta \\ &= \int\frac{2L_z d\theta}{\sqrt{L^2 - \frac{L_z^2}{\sin^2\theta}}\ \sin^2\theta} = -\arctan\frac{L_z\cos\theta}{\sqrt{L^2\sin^2\theta - L_z^2}} + \phi_0. \end{aligned}\tag{33}$$

ϕ_0 is an integration constant. This equation can be simplfied to

$$\cos^2\theta = \frac{(L^2 - L_z^2)\tan^2(\phi-\phi_0)}{L_z^2 + L^2\tan^2(\phi-\phi_0)}. \tag{34}$$

As ϕ is varied from 0 to 2π, $\cos\theta$ changes between $\pm\sqrt{L^2 - L_z^2}/L$. When $L_z = L$, the maximum possible value, $\cos\theta = 0$ and hence the motion is confined in the xy plane. As L_z decreases, the orbit is no longer confined in the xy plane, but still the polar angle varies uniquely as the azimuth is varied: a closed orbit. The $L_z \to 0$ limit is singular in this expression. The azimuth ϕ is then restricted to ϕ_0, while $\cos\theta$ can change for the entire range $[-1, 1]$. This behavior is exactly what we expect from closed orbits with fixed angular momenta.

The Hamilton–Jacobi equation for the remaining radial piece is

$$\frac{1}{2m}\left(\frac{dS_2}{dr}\right)^2 + \frac{L^2}{2mr^2} + V(r) = E. \tag{35}$$

Writing it

$$\frac{dS_2}{dr} = \sqrt{2mE - 2mV(r) - \frac{L^2}{r^2}} \ , \tag{36}$$

the problem is reduced to a matter of an integral. Therefore, Hamilton–Jacobi equation reduces the problem of three-dimensional motion down to a single integral, a dramatic simplification. The time-dependence of the motion is then obtained by the inverse Legendre transformation,

$$t = \frac{\partial S_2}{dE} = \int \frac{mdr}{\sqrt{2mE - 2mV(r) - \frac{L^2}{r^2}}}. \tag{37}$$

1.5 Kelper Motion

Let us apply the Hamilton–Jacobi equation to the Kepler motion. The only difference from the general case studied in the previous section is that we have a specific form of the potential

$$V(r) = -\frac{GMm}{r}. \tag{38}$$

Then the Hamilton–Jacobi equation remaining to be solved is

$$\frac{1}{2m}\left(\frac{dS_2}{dr}\right)^2 + \frac{L^2}{2mr^2} - \frac{GMm}{r} = E. \tag{39}$$

Solving this equation is straight-forward. Writing it

$$\frac{dS_2}{dr} = \sqrt{2mE + \frac{2GMm^2}{r} - \frac{L^2}{r^2}}, \tag{40}$$

even Mathematica can do this integral.

In fact, if what we want is the orbit, the integration is even simpler. Note first that we can always choose the z-axis such that the Kepler motion is in the x-y plane. Then $\sin\theta = 1$ and $L = L_z$. In this case, we find ϕ by

$$\phi = -\frac{\partial S_2}{\partial L} = L \int \frac{dr}{r\sqrt{2mEr^2 + 2GMm^2 r - L^2}}. \tag{41}$$

The integral can be evaluated to be

$$\phi = \arccos \frac{L^2 - GMm^2 r}{r\sqrt{2mEL^2 + G^2M^2m^4}} + \phi_0. \tag{42}$$

Or even better,

$$\left(GMm^2 + \sqrt{G^2M^2m^4 + 2mEL^2}\ \cos(\phi - \phi_0)\right) r = L^2. \tag{43}$$

Compared to the general formula for conic sections $r = ed/(1 - e\cos\theta)$ in the polar coordinate, we find

$$e = \frac{\sqrt{G^2M^2m^4 + 2mEL^2}}{GMm^2}, \tag{44}$$

$$d = L^2 \frac{GMm^2}{\sqrt{G^2M^2m^4 + 2mEL^2}}, \tag{45}$$

by choosing $\phi_0 = \pi$. An ellipse ($e < 1$), a parabola ($e = 1$) and a hyperbola ($e > 1$), is obtained depending on $E < 0$ (bound state), $E = 0$, and $E > 0$.[1]

1.6 Bohr-Sommerfeld Quantization Condition

It is interesting that this is the formalism with which Bohr and Sommerfeld came up with their quantization condition. They required that the action integral for a periodic motion must be integer multiples of $h = 2\pi\hbar$ for each degree of freedom.

Let us apply their condition to the motion in a central potential. First, the requirement that $\oint d\phi (dS(\phi)/d\phi) = 2\pi L_z = 2\pi m_l \hbar$ means the quantization of the angular momentum $L_z = m_l\hbar$.

The next one is

$$\begin{aligned} S_3 &= \oint \sqrt{L^2 - \frac{L_z^2}{\sin^2\theta}}\, d\theta \\ &= -2\pi(L - L_z). \end{aligned} \tag{46}$$

Then the quantization condition requires $2\pi(L - L_z) = \pi(L - m_l\hbar) = 2\pi m$ for $m \in \mathbb{Z}$, and hence $L = \hbar(m + m_l) \in \mathbb{Z}$. We normally write $L = l\hbar$. Because of the requirement $L^2 \geq L_z^2$, we also find $l \geq |m|$.

A more complicated condition for $S(r) = \oint dr(dS/dr)$ for $E = -|E| < 0$ yields

$$\oint dr \frac{dS_2}{dr} = 2\left(-L\pi + \frac{Gm^{3/2}M\pi}{\sqrt{2|E|}}\right) = 2\pi n'\hbar, \tag{47}$$

[1] See http://mathworld.wolfram.com/ConicSection.html for an introduction to conics.

together with $L_z = l\hbar$, leads to

$$E = -\frac{G^2 m^3 M^2}{2(n' + l)^2 \hbar^2}, \tag{48}$$

nothing but the energy levels of hydrogen-like atoms (replace GMm by Ze^2).

Why does this ad-hoc condition work to yield the exact result? This remains a mystery to this date. The WKB method shows that, in the limit of large quantum numbers, the conditions are more like the action being $(n + \frac{1}{2})h$ instead of nh. Of course for large n, the difference is negligible and they are consistent. It was a pure luck that this result was exact, even though it is supposed to be good for large quantum numbers. If you apply the same requirement to the harmonic oscillator, you find the energy levels to be $n\hbar\omega$ without the zero-point energy.

A Some Integrals

This appendix is just mathematical technicalities on how to do some of the integrals in this note.

The integral for the harmonic oscillator is done with the standard technique, to deal with a quadratic function in a square root.

$$\tilde{S} = \int \sqrt{2mE - m^2\omega^2 q^2}\, dq \tag{49}$$

We look for the change of variable to make the terms inside the square root to become $1 - \sin^2\theta = \cos^2\theta$ so that we can take the square root. Indeed, by defining $q = \sin\theta\sqrt{2mE}/m\omega$, we find

$$\tilde{S} = \int \sqrt{2mE - 2mE\sin^2\theta}\frac{\sqrt{2mE}}{m\omega}\cos\theta d\theta. \tag{50}$$

Taking the square root (of course, depending on the region of θ, the sign may be the opposite),

$$\tilde{S} = \int \sqrt{2mE}\cos\theta\frac{\sqrt{2mE}}{m\omega}\cos\theta d\theta = \frac{2E}{\omega}\int\frac{1+\cos 2\theta}{2}\theta = \frac{E}{\omega}\left(\theta + \frac{1}{2}\sin 2\theta\right). \tag{51}$$

The rest of the job is to rewrite θ in terms of q,

$$\tilde{S} = \frac{E}{\omega}\left(\arcsin\frac{m\omega q}{\sqrt{2mE}} + \frac{m\omega q}{\sqrt{2mE}}\sqrt{1-\left(\frac{m\omega q}{\sqrt{2mE}}\right)^2}\right). \tag{52}$$

One of the not-so-easy integrals is that for θ in Eq. (32). We need to compute

$$S_3(\theta) = \int\sqrt{L^2 - \frac{L_z^2}{\sin^2\theta}}d\theta = \int\frac{\sqrt{L^2 - L_z^2 - L^2\cos^2\theta}}{\sin\theta}d\theta. \tag{53}$$

First, we change the variable to $x = \cos\theta$,

$$S_3(\theta) = -\int\frac{\sqrt{L^2 - L_z^2 - L^2x^2}}{1-x^2}dx. \tag{54}$$

Then, to simplify the numerator, we make another change of variable $x = \frac{\sqrt{L^2-L_z^2}}{L}\sin\phi$ just as in the harmonic oscillator case,

$$S_3(\theta) = -\frac{L^2-L_z^2}{L}\int\frac{\cos^2\phi}{1-\frac{L^2-L_z^2}{L^2}\sin^2\phi}d\phi. \tag{55}$$

This is now a rational function of trigonometric functions, which can always be done analytically. The standard trick is to use the variable $t = \tan\phi$ (this works when the rational function depends only on even powers of trigonometric functions; otherwise you use $t = \tan(\phi/2)$ instead) so that

$$\cos^2\phi = \frac{1}{1+t^2}, \qquad \sin^2\phi = \frac{t^2}{1+t^2}, \qquad d\phi = \frac{dt}{1+t^2}. \tag{56}$$

Putting them together, we find

$$S_3(\theta) = -\frac{L^2-L_z^2}{L}\int\frac{\frac{1}{1+t^2}}{1-\frac{L^2-L_z^2}{L^2}\frac{t^2}{1+t^2}}\frac{dt}{1+t^2} = -\frac{L(L^2-L_z^2)}{L_z^2}\int\frac{dt}{(1+t^2)(\frac{L^2}{L_z^2}+t^2)}. \tag{57}$$

Using the partial fraction decomposition,

$$S_3(\theta) = -L\int\left(\frac{1}{1+t^2} - \frac{1}{\frac{L^2}{L_z^2}+t^2}\right)dt. \tag{58}$$

Both terms in the parentheses can be integrated with the standard formula, and we find

$$S_3(\theta) = -L\left(\arctan t - \frac{L_z}{L}\arctan\frac{L_z}{L}t\right). \tag{59}$$

The final job is to rewrite t in terms of the original variable θ. First going back to ϕ,

$$S_3(\theta) = -L\left(\phi - \frac{L_z}{L}\arctan\frac{L_z}{L}\tan\phi\right). \tag{60}$$

Now we try to relate ϕ to θ. Because $x = \cos\theta = \frac{\sqrt{L^2-L_z^2}}{L}\sin\phi$, we find

$$\cos\theta = \sqrt{1 - \frac{L^2-L_z^2}{L^2}\sin^2\phi} = \frac{\sqrt{L^2\sin^2\theta - L_z^2}}{\sqrt{L^2-L_z^2}}. \tag{61}$$

Therefore,

$$\tan\phi = \frac{L\cos\theta}{\sqrt{L^2\sin^2\theta - L_z^2}}. \tag{62}$$

We finally obtain

$$\begin{aligned} S_3(\theta) &= -L\left(\arctan\frac{L\cos\theta}{\sqrt{L^2\sin^2\theta - L_z^2}} - \frac{L_z}{L}\arctan\frac{L_z\cos\theta}{\sqrt{L^2\sin^2\theta - L_z^2}}\right) \\ &= -L\arctan\frac{L\cos\theta}{\sqrt{L^2\sin^2\theta - L_z^2}} + L_z\arctan\frac{L_z\cos\theta}{\sqrt{L^2\sin^2\theta - L_z^2}}. \end{aligned} \tag{63}$$

You can let Mathematica check easily that the $dS_3/d\theta$ is indeed $\sqrt{L^2 - L_z^2/\sin^2\theta}$. Note, however, that the integration constant can depend on L and L_z, which can play the role of initial conditions. Also, dS_3/dL in Eq. (33) calculated by differentiating by L first and integrating over θ later, is obtained straightforwardly from the above expression, except that you need to allow the integration constant $L_z\phi_0$ to $S_3(\theta)$.

Another use of Eq. (63) is the Bohr–Sommerfeld quantization condition. By requiring that S_3 for a period of the motion is an integer multiple of h, we obtain quantization condition for L. A period is given by arctan changing from $\pi/2$ to $-\pi/2$, and then back to $\pi/2$. Therefore, S_3 for a period of the motion is

$$S_3 = -2\pi L + 2\pi L_z. \tag{64}$$

Therefore, $L - L_z$ must be an integer multiple of $\hbar$. On the other hand, the same condition for $S_4(\phi) = 2\pi L_z$ requires that $L_z = m\hbar$, $m \in \mathbb{Z}$. Therefore, L itself must be an integer multiple of $\hbar$, $L = l\hbar$, $l \in \mathbb{Z}$. This was the argument by Bohr and Sommerfeld why angular momentum is quantized.

Finally, the integral for the Kepler motion is

$$S_2(r) = \int \sqrt{2mE + \frac{2GMm^2}{r} - \frac{L^2}{r^2}} dr = \int \sqrt{2mEr^2 + 2GMm^2 r - L^2} \frac{dr}{r}. \tag{65}$$

If $E < 0$ for elliptic orbits, the terms in the square root can be rewritten as

$$S_2(r) = \int \sqrt{\frac{G^2M^2m^3}{2|E|} - L^2 - 2m|E| \left(r - \frac{GMm}{2|E|} \right)^2} \frac{dr}{r}. \tag{66}$$

Changing the variable to $r' = r - \frac{GMm}{2|E|}$,

$$S_2(r) = \int \sqrt{\frac{G^2M^2m^3}{2|E|} - L^2 - 2m|E|r'^2} \frac{dr'}{r' + \frac{GMm}{2|E|}}. \tag{67}$$

To open the square root, the next change of variable is

$$(2m|E|)^{1/2} r' = \left(\frac{G^2M^2m^3}{2|E|} - L^2 \right)^{1/2} \sin\eta. \tag{68}$$

Then the integral becomes

$$S_2(r) = \left(\frac{G^2M^2m^3}{2|E|} - L^2 \right)^{1/2} \int \frac{\cos^2\eta d\eta}{\frac{GMm}{2|E|} + \left(\left(\frac{GMm}{2|E|} \right)^2 - \frac{L^2}{2m|E|} \right)^{1/2} \sin\eta}. \tag{69}$$

To integrate a rational function of η, we use the conventional change of variable

$$t = \tan\frac{\eta}{2}, \qquad \cos\eta = \frac{1-t^2}{1+t^2}, \qquad \sin\eta = \frac{2t}{1+t^2}, \qquad d\eta = \frac{2dt}{1+t^2}. \tag{70}$$

Then the t integral can be done after a partial fraction decomposition. We find

$$S_2(r) = \left(\frac{G^2M^2m^3}{2|E|} - L^2 \right)^{1/2} \left(\frac{a}{b}\eta - \frac{2\sqrt{a^2-b^2}}{b} \arctan\frac{b\cos\frac{\eta}{2} + a\sin\frac{\eta}{2}}{\sqrt{a^2-b^2}\cos\frac{\eta}{2}} + \cos\eta \right), \tag{71}$$

with

$$a = \frac{GMm}{2|E|}, \qquad b = \left(\left(\frac{GMm}{2|E|} \right)^2 - \frac{L^2}{2m|E|} \right)^{1/2}. \tag{72}$$

The integral for a period Eq. (47) is obtained by varying η from 0 to 2π.