Dirac Delta Function

1 Definition

Dirac's delta function is defined by the following property

$$\delta(t) = \begin{cases} 0 & t \neq 0 \\ \infty & t = 0 \end{cases} \tag{1}$$

with

$$\int_{t_1}^{t_2} dt \delta(t) = 1 \tag{2}$$

if $0 \in [t_1, t_2]$ (and zero otherwise). It is "infinitely peaked" at $t = 0$ with the total area of unity. You can view this function as a limit of Gaussian

$$\delta(t) = \lim_{\sigma \to 0} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-t^2/2\sigma^2} \tag{3}$$

or a Lorentzian

$$\delta(t) = \lim_{\epsilon \to 0} \frac{1}{\pi} \frac{\epsilon}{t^2 + \epsilon^2}. \tag{4}$$

The important property of the delta function is the following relation

$$\int dt f(t) \delta(t) = f(0) \tag{5}$$

for any function $f(t)$. This is easy to see. First of all, $\delta(t)$ vanishes everywhere except $t = 0$. Therefore, it does not matter what values the function $f(t)$ takes except at $t = 0$. You can then say $f(t)\delta(t) = f(0)\delta(t)$. Then $f(0)$ can be pulled outside the integral because it does not depend on t, and you obtain the r.h.s. This equation can easily be generalized to

$$\int dt f(t) \delta(t - t_0) = f(t_0). \tag{6}$$

Mathematically, the delta function is not a function, because it is too singular. Instead, it is said to be a "distribution." It is a generalized idea of functions, but can be used only inside integrals. In fact, $\int dt \delta(t)$ can be regarded as an "operator" which pulls the value of a function at zero. Put it this way, it sounds perfectly legitimate and well-defined. But as long as it is understood that the delta function is eventually integrated, we can use it as if it is a function. One caveat is that you are not allowed to multiply delta functions whose arguments become simultaneously zero, *e.g.*, $\delta(t)^2$. If you try to integrate it over t, you would obtain $\delta(0)$, which is infinite and does not make sense. But physicists are sloppy enough to even use $\delta(0)$ sometimes, as we will discuss below.

2 Fourier Transformation

It is often useful to talk about Fourier transformation of functions. For a function $f(t)$, you define its Fourier transform

$$\tilde{f}(s) \equiv \int_{-\infty}^{\infty} dt \frac{e^{its}}{\sqrt{2\pi}} f(t). \tag{7}$$

This transform is reversible, *i.e.*, you can go back from $\tilde{f}(s)$ to $f(t)$ by

$$f(t) = \int_{-\infty}^{\infty} ds \frac{e^{-its}}{\sqrt{2\pi}} \tilde{f}(s). \tag{8}$$

You may recall that the patterns from optical or X-ray diffraction are Fourier transforms of the structure. For example, Laue determined the crystallographic structure of solid by doing inverse Fourier-transform of the X-ray diffraction patterns.

If you set $f(t) = \delta(t)$ in the above equations, you find

$$\tilde{\delta}(s) \equiv \int_{-\infty}^{\infty} dt \frac{e^{its}}{\sqrt{2\pi}} \delta(t) = \frac{1}{\sqrt{2\pi}}, \tag{9}$$

$$\delta(t) = \int_{-\infty}^{\infty} ds \frac{e^{-its}}{\sqrt{2\pi}} \frac{1}{\sqrt{2\pi}} = \int_{-\infty}^{\infty} ds \frac{e^{-its}}{2\pi}. \tag{10}$$

In other words, the delta function and a constant $1/\sqrt{2\pi}$ are Fourier-transform of each other.

Another way to see the integral representation of the delta function is again using the limits. For example, using the limit of the Gaussian Eq. (3),

$$\begin{aligned}
\delta(t) &= \lim_{\sigma \to 0} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-t^2/2\sigma^2} \\
&= \lim_{\sigma \to 0} \int_{-\infty}^{\infty} d\omega \frac{1}{2\pi} e^{-\omega^2\sigma^2/2} e^{-i\omega t} \\
&= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t}. \qquad (11)
\end{aligned}$$

3 Position Space

Dirac invented the delta function to deal with the completeness relation for position and momentum eigenstates. The eigenstate for the position operator x

$$x|x'\rangle = x'|x'\rangle \tag{12}$$

must be normalized in a way that the analogue of the completeness relation holds for discrete eigenstates $1 = \sum_a |a\rangle\langle a|$. Because the eigenvalues of the position operator are continuous, the sum is replaced by an integral

$$1 = \int |x'\rangle dx' \langle x'|. \tag{13}$$

For the case of the discrete eigenstates, using the completeness relationship twice gives a consistent result because of the orthonomality of the eigenstates $\langle a'|a''\rangle = \delta_{a',a''}$:

$$\begin{aligned} 1 &= 1 \times 1 = \left(\sum_{a'} |a'\rangle\langle a'| \right) \left(\sum_{a''} |a''\rangle\langle a''| \right) \\ &= \sum_{a',a''} |a'\rangle(\langle a'|a''\rangle)\langle a''| \\ &= \sum_{a',a''} |a'\rangle\delta_{a',a''}\langle a''| \\ &= \sum_{a'} |a'\rangle\langle a'| = 1. \end{aligned} \tag{14}$$

Therefore, we need also the states $|x'\rangle$ to be orthonomal. To see it, we try the same thing as in the discrete spectrum

$$\begin{aligned} 1 &= 1 \times 1 = \left(\int |x'\rangle dx' \langle x'| \right) \left(\int |x''\rangle dx'' \langle x''| \right) \\ &= \int dx' dx'' |x'\rangle(\langle x'|x''\rangle)\langle x''|. \end{aligned} \tag{15}$$

Now we can determine what the "orthonomality" condition must look like. Only by setting $\langle x'|x'' = \delta(x' - x'')$, we find

$$\begin{aligned} 1 &= \int dx' dx'' |x'\rangle\delta(x' - x'')\langle x''| \\ &= \int dx' |x'\rangle\langle x'| = 1. \end{aligned} \tag{16}$$

At the last step, I used the property of the delta function that the integral over x'' inserts the value $x'' = x'$ into the rest of the integrand. This is why we need the "delta-function normalization" for the position eigenkets.

It is also worthwhile to note that the delta function in position has the dimension of $1/L$, because its integral over the position is unity. Therefore the position eigenket $|x'\rangle$ has the dimension of $L^{-1/2}$.

4 Momentum Space

As you see in Sakurai Eq. (1.7.32), the eigenstates of the position and momentum operators have the inner product

$$\langle x'|p'\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{ip'x'/\hbar} \tag{17}$$

From this expression, you can see that the wave functions in the position space and the momentum space are related by the Fourier-transform.

$$\begin{aligned} \phi_\alpha(p') &= \langle p'|\alpha\rangle \\ &= \int \langle p'|x'\rangle dx' \langle x'|\alpha\rangle \\ &= \int dx' \frac{e^{-ip'x'/\hbar}}{\sqrt{2\pi\hbar}} \psi_\alpha(x'). \end{aligned} \tag{18}$$

The completeness of the momentum eigenstates can also be shown using the properties of the delta function.

$$\begin{aligned} \int |p'\rangle dp' \langle p'| &= \int dp' dx' dx'' |x'\rangle \langle x'|p'\rangle \langle p'|x''\rangle \langle x''| \\ &= \int dp' dx' dx'' |x'\rangle \frac{e^{ix'p'/\hbar}}{\sqrt{2\pi\hbar}} \frac{e^{-ix''p'/\hbar}}{\sqrt{2\pi\hbar}} \langle x''| \\ &= \int dx' dx'' |x'\rangle \langle x''| \int dp' \frac{e^{i(x'-x'')p'/\hbar}}{2\pi\hbar}. \end{aligned} \tag{19}$$

The last integral, after changing the variable from p' to $k = p/\hbar$, is nothing but the Fourier-integral expression for the delta function. Therefore,

$$\begin{aligned} &= \int dx' dx'' |x'\rangle \langle x''| \delta(x' - x'') \\ &= \int dx' |x'\rangle \langle x'| = 1. \end{aligned} \tag{20}$$

This proves the completeness of the momentum eigenstates.