Midterm Exam (221A), due Oct 27, 4pm

1. A particle of mass m is allowed to move only along the circle of radius R on a plane, $x = R\cos\theta$, $y = R\sin\theta$. [30]

   (a) Show that the Lagrangian is $L = \frac{m}{2}R^2\dot{\theta}^2$, and write down the canonical momentum p_θ and the Hamiltonian. [5]

   (b) Write down the Heisenberg equation of motion, and solve them. (So far no representation was taken.) [5]

   (c) Write down the normalized position-space wave function $\psi_k(\theta) = \langle\theta|k\rangle$ for the momentum eigenstates $p_\theta|k\rangle = k\hbar|k\rangle$, and show that only $k = n \in \mathbb{Z}$ are allowed because of the requirement $\psi(\theta + 2\pi) = \psi(\theta)$. [5]

   (d) Show the orthonormality $\langle n|m\rangle = \int_0^{2\pi} \psi_n^* \psi_m d\theta = \delta_{n,m}$. [5]

   (e) Now we introduce a constant magnetic field B inside the radius $r < d < R$ but no magnetic field outside $r > d$, with the vector potential is

   $$(A_x, A_y) = \begin{cases} \frac{B}{2}(-y, x) & (r < d) \\ \frac{B}{2}\frac{d^2}{r^2}(-y, x) & (r > d). \end{cases} \tag{1}$$

   Write the Lagrangian, derive the Hamiltonian, and show that energy eigenvalues are influenced by the magnetic field even though the particle does not "see" the magnetic field directly. [10]

2. Consider a charged particle on the x-y plane in a constant magnetic field $\vec{B} = (0, 0, B)$ with the Hamiltonian (assume $eB > 0$) [45]

   $$H = \frac{\Pi_x^2 + \Pi_y^2}{2m}, \qquad \Pi_i = p_i - \frac{e}{c}A_i. \tag{2}$$

   (a) Use the so-called "symmetric gauge" $\vec{A} = \frac{B}{2}(-y, x)$, and simplify the Hamiltonian using the two annihilation operators a_x, a_y for a suitable choice of ω. [5]

   (b) Further define $a_z = \frac{1}{2}(a_x + i a_y)$, $a_{\bar{z}} = \frac{1}{2}(a_x - i a_y)$, and rewrite the Hamiltonian using them. General states are given in the form

   $$|n, m\rangle = \frac{(a_z^\dagger)^n}{\sqrt{n!}} \frac{(a_{\bar{z}}^\dagger)^m}{\sqrt{m!}} |0, 0\rangle \tag{3}$$

   starting from the ground state $a_z|0,0\rangle = a_{\bar{z}}|0,0\rangle = 0$. Show that they are Hamiltonian eigenstates of energies $\hbar\omega(2n+1)$. [5]

   (c) For an electron, what is the excitation energy under $B = 100$ kG? [5]

(d) Work out the wave function $\langle x, y|0, 0\rangle$ in the position space. [5]

(e) $|0, m\rangle$ are all ground states. Show that their position-space wave functions are given by

$$\psi_{0,m}(z, \bar{z}) = N z^m e^{-eB\bar{z}z/4\hbar c}, \tag{4}$$

where $z = x + iy$, $\bar{z} = x - iy$. Determine N. [5]

(f) Plot the probability density of the wave function for $m = 0$, 3, and 10 (use `ContourPlot` or `Plot3D`) on the same scale. [5]

(g) Assuming that the system is a circle of a finite radius R, show that there are only a finite number of ground states. Work out the number approximately for a large R. [5]

(h) Show that the coherent state $e^{f a_z^\dagger}|0, 0\rangle$ represents a near-classical cyclotron motion in the position space. [10]

3. Read the article W.-T. Lee *et al*, "Observation of Scalar Ahoronov–Bohm Effect with Longitudinally Polarized Neutrons," *Phys. Rev. Lett.* **80**, 3165 (1998), which realized the gedanken experiment Sakurai discusses in pp. 123–125. Show that the change in the count rate in Fig. 5 is what is expected theoretically. The magnetic moment of the neutron can be found, *e.g.*, from Particle Data Group at Lawrence Berkeley National Laboratory. [10]

4. Read the article T. Araki *et al*, "Measurement of Neutrino Oscillation with KamLAND: Evidence of Spectral Distortion," *Phys. Rev. Lett.* **94**, 081801 (2005), which shows the neutrino oscillation, a quantum phenomenon demonstrated at the largest distance scale yet (about 180 km). [20]

(a) The Hamiltonian for an ultrarelativistic particle is approximated by

$$H = \sqrt{p^2c^2 + m^2c^4} \simeq pc + \frac{m^2c^3}{2p}, \tag{5}$$

for $p = |\vec{p}|$. Suppose in a basis of two states, m^2 is given as a two-by-two matrix

$$m^2 = m_0^2 I + \frac{\Delta m^2}{2} \begin{pmatrix} -\cos 2\theta & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix}. \tag{6}$$

Write down the eigenstates of m^2. [5]

(b) Calculate the probability for the state $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ to be still found in the same state after time interval t for a definite momentum p. [5]

(c) Using the data shown in Fig. 3, estimate approximately values of Δm^2 and $\sin^2 2\theta$. [5]