221A Lecture Notes
Electromagnetic Couplings

1 Classical Mechanics

The coupling of the electromagnetic field with a charged point particle of charge e is given by a term in the action (MKSA system)

$$S_{int} = \int L_{int} dt = -e \int A_\mu dx^\mu = -e \int (A^0 c dt + \vec{A} \cdot d\vec{x}) = \int \left(-e\phi + e\vec{A} \cdot \dot{\vec{x}} \right) dt. \tag{1}$$

Here, the four-vector notation is $A^\mu = (\frac{1}{c}\phi, \vec{A})$ where ϕ is the scalar potential, and $dx^\mu = (cdt, d\vec{x})$. The Lorentz index contraction is done as $a_\mu b^\mu = a^\mu b_\mu = a^0 b^0 - \vec{a} \cdot \vec{b}$.

Why is this term the right start? There are various ways to introduce it depending on what you are familiar with. Below, I present two ways to introduce this coupling.

1.1 Electromagnetic field's story

One way to introduce the coupling Eq. (1) is by starting with Maxwell's equations. In the MKSA system, they read

$$\begin{aligned} \vec{\nabla} \cdot \vec{E} &= \frac{1}{\epsilon_0} \rho, & (2) \\ \vec{\nabla} \times \vec{B} &= \frac{1}{c^2} \dot{\vec{E}} + \mu_0 \vec{\jmath}, & (3) \\ \vec{\nabla} \times \vec{E} &= -\dot{\vec{B}}, & (4) \\ \vec{\nabla} \cdot \vec{B} &= 0. & (5) \end{aligned}$$

The electric field is denoted by $\vec{E}$, the magnetic flux density by $\vec{B}$, the charge density by ρ, the current density by $\vec{\jmath}$.

Maxwell's equations appear *overdetermined*, namely that there are eight equations (one each for Eqs. (2,5) and three components each for Eqs. (3,4)) for six variables (three each for $\vec{E}$ and $\vec{B}$). This doesn't turn out to be a problem because the last two equations are trivial once expressed in terms of the scalar and vector potentials

$$\vec{E} = -\vec{\nabla}\phi - \dot{\vec{A}}, \tag{6}$$
$$\vec{B} = \vec{\nabla} \times \vec{A}, \tag{7}$$

since

$$\vec{\nabla} \times \vec{E} + \dot{\vec{B}} = -\vec{\nabla} \times \vec{\nabla}\phi - \vec{\nabla} \times \dot{\vec{A}} + \vec{\nabla} \times \dot{\vec{A}} = 0, \tag{8}$$
$$\vec{\nabla} \cdot \vec{B} = \vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0. \tag{9}$$

The first two equations are rewritten as

$$\epsilon_0(\Delta\phi + \vec{\nabla} \cdot \dot{\vec{A}}) = -\rho, \tag{10}$$
$$c^2\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = -\vec{\nabla}\dot{\phi} - \ddot{\vec{A}} + \frac{1}{\epsilon_0}\vec{\jmath}. \tag{11}$$

In the last term, I used the identity $\epsilon_0\mu_0 = 1/c^2$. This way, we have four equations for four unknowns, perfect differential equations. These equations can be derived from the action

$$S = \int dt d^3x \left[\frac{\epsilon_0}{2}\left(\vec{E}^2 - c^2\vec{B}^2\right) - \phi\rho + \vec{A} \cdot \vec{\jmath}\right], \tag{12}$$

if it is viewed as a function of the scalar potential ϕ and the vector potential $\vec{A}$ as independent variables. Indeed, using the expressions of $\vec{E}$ and $\vec{B}$ in terms of potentials, the action is

$$S = \int dt d^3x \left[\frac{\epsilon_0}{2}\left((\vec{\nabla}\phi + \dot{\vec{A}})^2 - c^2(\vec{\nabla} \times \vec{A})^2\right) - \phi\rho + \vec{A} \cdot \vec{\jmath}\right], \tag{13}$$

and its variation with respect to ϕ and $\vec{A}$ is

$$\begin{aligned} S &= \int dt d^3x \left[\epsilon_0\left(-\delta\phi\Delta\phi - \delta\phi\vec{\nabla} \cdot \dot{\vec{A}} - \delta\vec{A} \cdot \vec{\nabla}\dot{\phi} - \delta\vec{A} \cdot \ddot{\vec{A}}\right.\right. \\ &\qquad \left.\left. - c^2\delta\vec{A} \cdot \vec{\nabla} \times (\vec{\nabla} \times \vec{A})\right) - \delta\phi\rho + \delta\vec{A} \cdot \vec{\jmath}\right], \end{aligned} \tag{14}$$

up to surface terms, reproducing the Maxwell's equations.

Then suppose that the charge density and the current density are given by a collection of particles of charges q_i at positions $\vec{x}_i(t)$

$$\rho(\vec{x}) = \sum_i q_i\delta(\vec{x} - \vec{x}_i), \tag{15}$$
$$\vec{\jmath}(\vec{x}) = \sum_i \dot{\vec{x}}_i q_i\delta(\vec{x} - \vec{x}_i). \tag{16}$$

Then the terms in the action where the electromagnetic fields couple to the charge and current densities become

$$\int dt d^3x \left[-\phi\rho + \vec{A}\cdot\vec{j}\right] = \sum_i q_i \int dt \left[-\phi(\vec{x}_i) + \vec{A}(\vec{x}_i)\cdot\dot{\vec{x}}_i\right]. \tag{17}$$

Keeping only one particle of charge e, we obtain the coupling Eq. (1).

1.2 Particle's story

Another way to justify the Lagrangian is to look at the particle side of the story. We want the equation of motion for a charged particle to be

$$m\ddot{\vec{x}} = q\vec{E} + q\vec{v}\times\vec{B}. \tag{18}$$

Starting from the action

$$S = \int dt \left(\frac{m}{2}\dot{\vec{x}}^2 - q\phi + q\vec{A}\cdot\vec{v}\right), \tag{19}$$

we derive the equation of motion. The Lagrangian is the integrand, and

$$\frac{\partial L}{\partial \dot{x}^i} = m\dot{x}^i + qA^i, \tag{20}$$

$$\frac{\partial L}{\partial x^i} = -q\nabla_i\phi + q\nabla_i A^j v^j. \tag{21}$$

Therefore,

$$\begin{aligned}
0 = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} - \frac{\partial L}{\partial \dot{x}^i} &= \frac{d}{dt}(m\dot{x}^i + qA^i) + q\nabla_i\phi - q\nabla_i A^j v^j \\
&= m\ddot{x}^i + q(v^j\nabla_j A^i + \dot{A}^i) + q\nabla_i\phi - qv^j\nabla_i A^j \\
&= m\ddot{x}^i + q(\dot{A}^i + \nabla_i\phi) - qv^j\epsilon_{ijk}B_k \\
&= \left[m\ddot{\vec{x}} - q\vec{E} - q(\vec{v}\times\vec{B})\right]_i. \qquad (22)
\end{aligned}$$

This reproduces the equation of motion of a particle in the electromagnetic field Eq. (18). The only tricky part in this algebra is the time derivative of the vector potential,

$$\frac{d}{dt}\vec{A}(\vec{x}(t), t) = v^j\nabla_j A^i + \dot{A}^i. \tag{23}$$

1.3 Gaussian System

In quantum mechanics (or for this matter in many textbooks and papers), a different unit system called Gaussian unit is often used, where the Maxwell's equations become

$$\vec{\nabla} \cdot \vec{E} = 4\pi\rho, \tag{24}$$

$$\vec{\nabla} \times \vec{B} = \frac{1}{c}\dot{\vec{E}} + \frac{4\pi}{c}\vec{\jmath}, \tag{25}$$

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\dot{\vec{B}}, \tag{26}$$

$$\vec{\nabla} \cdot \vec{B} = 0. \tag{27}$$

This is what Sakurai uses in his book without saying so.[1] This unit system can be obtained from the MKSA system by choosing a new unit for the charge such that $4\pi\epsilon_0 = 1$, and changing the normalization of the vector potential by c. Note that the Gaussian unit is used conventionally together with the cgs system.

This is done first by setting the dielectric constant of the vacuum

$$4\pi\epsilon_0 = 1. \tag{28}$$

Or, equivalently, $\mu_0 = 4\pi/c^2$. Why are we allowed to do this? Well, the "A" part of the MKSA system is defined as follows: there is a force of 2×10^{-7}N between two parallel currents of 1A per 1m separated by 1m. The force is

$$F = \frac{\mu_0}{2\pi r} I I' l, \tag{29}$$

where I, I' are two electric currents, r the distance between them and l the length for which the force acts. Note that only the combination $\mu_0 I I'$ matters. In other words, I can choose the unit for the electric current such that $\mu_0 = 4\pi/c^2$. Once this is done, I don't need a separate unit for the electric current; I don't need "A" part of the MKSA. Then the force between currents is

$$F = \frac{2}{c^2 r} I I' l, \tag{30}$$

[1] Unfortunately, people use yet another unit system (rationalized unit) in quantum electrodynamics (relativistic quantum theory of photons and electrons), where $\epsilon_0 = 1$ and hence 4π does not appear in Maxwell's equation. The electric and magnetic fields also differ by a factor of $\sqrt{4\pi}$.

Because the combination II'/c^2 must have the unit of force (say, N or dimension MLT^{-2}), the electric current has the dimension of $M^{1/2}L^{3/2}T^{-2}$. Correspondingly, the electric charge $M^{1/2}L^{3/2}T^{-1}$. The fractional power may look very odd, but it is simply because force, energy, etc depend always on the *square* of the charges or currents. One advantage of this unit system is that the Coulomb potential from a point source becomes just

$$\phi = \frac{q}{r} \tag{31}$$

with no coefficients. This way of measuring the electric charge is called esu for electro-static unit.

Another change we do is to change the normalization of the vector potential $\vec{A}$ (and correspondingly $\vec{B}$) by a factor of c. This is for the purpose of making all components of the four-vector potential $A^\mu = (\phi, \vec{A})$ have the same dimension. (In MKSA, the four-vector potential is $A^\mu = (\frac{\phi}{c}, \vec{A})$.) Note that the Lorentz force is then

$$\vec{F} = \frac{q}{c}\vec{v} \times \vec{B} \tag{32}$$

with $1/c$ factor.

In the end, we find the action

$$\begin{aligned} S &= \int dt d^3x \left[\frac{1}{8\pi}\left(\vec{E}^2 - \vec{B}^2\right) - \phi\rho + \frac{1}{c}\vec{A}\cdot\vec{j}\right] \\ &= \int dt d^3x \left[-\frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu} - \frac{1}{c}A_\mu j^\mu\right], \end{aligned} \tag{33}$$

where $j^\mu = (c\rho, \vec{j})$. The four-component notation is $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ with $\partial_\mu = (\frac{1}{c}\frac{\partial}{\partial t}, \vec{\nabla})$.

Complaining that we are not sticking to the SI system? Well, we haven't gotten ridden of the yard-pound system either!^_^; On top of these changes, nuclear and particle physicists further simplify the unit sytem by setting $\hbar = c = 1$ (except that $\epsilon_0 = 1$ instead of $4\pi\epsilon_0$, namely it is "rationalized"), atomic physicists and some chemists instead set $m_e = \hbar = e = 1$. This is possible because we are still left with three basic units for mass, length, and time even after we set the dielectric constant to a number. Therefore we can specify up to three more fundamental constants to unity for convenience.

For the rest of the discussions, I use the Gaussian system introduced here to be consistent with Sakurai.

1.4 Gauge Invariance

The scalar and vector potentials are not directly observable, as you can change the "gauge." The gauge transformations are defined by a scalar function $\Lambda(\vec{x}, t)$,

$$\phi \to \phi' = \phi + \frac{1}{c}\dot{\Lambda}, \qquad \vec{A} \to \vec{A}' = \vec{A} - \vec{\nabla}\Lambda. \tag{34}$$

In the four-vector notation, it is $A^\mu \to A'^\mu = A^\mu + \partial^\mu \Lambda$. It is easy to verify that this transformation does not change the electromagnetic fields, because $F'^{\mu\nu} - F^{\mu\nu} = \partial^\mu \partial^\nu \Lambda - \partial^\nu \partial^\mu \Lambda = 0$. But the action seems to depend directly on the potentials. Why is this OK?

The change in the term Eq. (1) is

$$\begin{aligned}
\Delta S_{int} &= -e \int \frac{1}{c}\dot{\Lambda}(\vec{x}, t) dt - \frac{e}{c} \int \vec{\nabla}\Lambda(\vec{x}, t) \cdot \dot{\vec{x}} dt \\
&= -\frac{e}{c} \int dt \frac{d}{dt} \Lambda(\vec{x}(t), t) \\
&= -\frac{e}{c} (\Lambda(\vec{x}(t_f), t_f) - \Lambda(\vec{x}(t_i), t_i)).
\end{aligned} \tag{35}$$

Therefore, the change is a total derivative which depends only on the initial and final data. In classical mechanics, the equation of motion is all you care, and the total derivative term does not affect the equation of motion. Of course, this point can be readily checked by the fact that the equation of motion Eq. (18) involves only the electromagnetic fields, not potentials. But in quantum mechanics, the total derivative terms in the action matter because they act on the initial and final kets.

1.5 Canonical Momentum

The canonical momentum of the particle is defined as usual,

$$\vec{p} = \frac{\partial L}{\partial \dot{\vec{x}}} = m\dot{\vec{x}} + \frac{e}{c}\vec{A}. \tag{36}$$

In other words, the velocity of the particle is given by

$$\vec{v} = \dot{\vec{x}} = \frac{1}{m}\left(\vec{p} - \frac{e}{c}\vec{A}\right). \tag{37}$$

It is not intuitive that the "momentum" is not proportional to the velocity, but it isn't. Sakurai defines

$$\vec{\Pi} = \vec{p} - \frac{e}{c}\vec{A}, \tag{38}$$

and calls it kinematical or mechanical momentum. I call it kinetic momentum. Whatever the name is, the distinction from the canonical momentum must be made carefully and consistently.

The Hamiltonian is obtained from the standard procedure,

$$\begin{aligned} H &= \vec{p}\cdot\dot{\vec{x}} - L \\ &= \vec{p}\cdot\frac{1}{m}\Pi - \frac{1}{2m}\vec{\Pi}^2 + e\phi - \frac{e}{c}\vec{A}\cdot\frac{1}{m}\vec{\Pi} \\ &= \frac{1}{2m}\vec{\Pi}^2 + e\phi. \end{aligned} \tag{39}$$

The Hamilton equations of motion follow from this expression, except that you always have to talk about $\vec{x}$ and $\vec{p}$ being independent, satisfying the canonical Poisson bracket (commutation relation once you are in the quantum theory), even though the kinetic momentum is $\vec{\Pi} \neq \vec{p}$. We find

$$\frac{d}{dt}\vec{x} = \frac{\partial H}{\partial \vec{p}} = \frac{1}{m}\vec{\Pi}, \tag{40}$$

$$\frac{d}{dt}\vec{p} = -\frac{\partial H}{\partial \vec{x}} = \frac{e}{mc}\Pi^i\vec{\nabla}A^i - e\vec{\nabla}\phi. \tag{41}$$

Note that the canonical momentum is gauge-dependent, and the r.h.s. of the second Hamilton equation is also gauge-dependent. The only physical (i.e., gauge-invariant) combination is

$$\begin{aligned} \frac{d}{dt}\Pi^i &= \frac{d}{dt}p^i - \frac{e}{c}\left(\dot{A}^i + \dot{x}^j\nabla_j A^i\right) \\ &= \frac{e}{mc}\Pi^j\nabla_i A^j - e\nabla_i\phi - \frac{e}{c}\left(\dot{A}^i + \dot{x}^j\nabla_j A^i\right) \\ &= \frac{e}{c}(\dot{x}^j\nabla_i A^j - \dot{x}^j\nabla_j A^i) - e\nabla_i\phi - \frac{e}{c}\dot{A}^i \\ &= \frac{e}{c}\epsilon_{ijk}\dot{x}^j B_k + eE_i \\ &= \frac{e}{c}(\vec{v}\times\vec{B})_i + eE_i. \end{aligned} \tag{42}$$

Together with Eq. (40), we recover the Euler–Lagrange equation of motion

$$m\ddot{\vec{x}} = \frac{e}{c}(\vec{v}\times\vec{B}) + e\vec{E}. \tag{43}$$

2 Quantum Mechanics

2.1 Schrödinger Equation

Because of the Hamiltonian given in Eq. (39), the Schrödinger equation is

$$i\hbar\frac{\partial}{\partial t}\psi = \left[\frac{1}{2m}\left(\frac{\hbar}{i}\vec{\nabla} - \frac{e}{c}\vec{A}\right)^2 + e\phi\right]\psi. \tag{44}$$

One caution with this expression is how the derivative acts on quantities. The only term of concern is the first term in the square bracket,

$$\frac{1}{2m}\left(\frac{\hbar}{i}\vec{\nabla} - \frac{e}{c}\vec{A}\right)^2\psi = \frac{1}{2m}\left(\frac{\hbar}{i}\vec{\nabla} - \frac{e}{c}\vec{A}\right)\left(\frac{\hbar}{i}\vec{\nabla} - \frac{e}{c}\vec{A}\right)\psi, \tag{45}$$

where the derivative in the second parentheses acts only on ψ, while that in the first acts not only on ψ but also on $\vec{A}$ in the second parentheses, too.

The Heisenberg equation of motion is basically the same as the Hamilton equation of motion, if you work it out carefully by paying attention to the ordering of operators. The only caution is that the momentum operator does not commute with the electric nor magnetic fields. You find

$$\frac{d}{dt}\vec{x} = \frac{1}{i\hbar}[\vec{x}, H] = \frac{1}{m}\vec{\Pi}, \tag{46}$$

$$\frac{d}{dt}\vec{p} = \frac{1}{i\hbar}[\vec{p}, H] = \frac{e}{2mc}(\Pi^i(\vec{\nabla}A^i) + (\vec{\nabla}A^i)\Pi^i) - e\vec{\nabla}\phi. \tag{47}$$

Here you see that $\vec{\nabla}A^i$ and Π^i are symmetrized. Also the gauge-invariant combination is

$$\begin{aligned}\frac{d}{dt}\Pi^i &= \frac{1}{i\hbar}[\Pi^i, H] \\ &= \frac{e}{2c}\epsilon_{ijk}(\dot{x}^j B_k + B_k\dot{x}^j) + e\vec{E}_i. \end{aligned} \tag{48}$$

The Lorentz-force term is again symmetrized.

2.2 Gauge Invariance

Because the gauge transformation changes the action by a surface term, it requires the phase change of the initial and final kets. This is clear from the path integral expression,

$$\langle \vec{x}_f, t_f | \vec{x}_i, t_i \rangle = \int \mathcal{D}\vec{x}(t) \exp \frac{i}{\hbar} \int_{t_i}^{t_f} dt \left(\frac{m}{2} \dot{\vec{x}}^2 - e\phi + \frac{e}{c} \vec{A} \cdot \dot{\vec{x}} \right). \tag{49}$$

In the new gauge Eq. (34),

$$\begin{aligned}
&\int \mathcal{D}\vec{x}(t) \exp \frac{i}{\hbar} \int_{t_i}^{t_f} dt \left(\frac{m}{2} \dot{\vec{x}}^2 - e\phi' + \frac{e}{c} \vec{A}' \cdot \dot{\vec{x}} \right) \\
&= \int \mathcal{D}\vec{x}(t) \exp \frac{i}{\hbar} \left[\int_{t_i}^{t_f} dt \left(\frac{m}{2} \dot{\vec{x}}^2 - e\phi + \frac{e}{c} \vec{A} \cdot \dot{\vec{x}} \right) - \frac{e}{c} (\Lambda(\vec{x}(t_f), t_f) - \Lambda(\vec{x}(t_i), t_i)) \right] \\
&= \langle \vec{x}_f, t_f | \vec{x}_i, t_i \rangle e^{-i \frac{e}{\hbar c} (\Lambda(t_f) - \Lambda(t_i))}.
\end{aligned} \tag{50}$$

Therefore, defining a new phase convention for the position eigenstates

$$\langle \vec{x}_f, t_f |' = \langle \vec{x}_f, t_f | e^{-i \frac{e}{\hbar c} \Lambda(t_f)}, \qquad | \vec{x}_i, t_i \rangle' = | \vec{x}_i, t_i \rangle e^{i \frac{e}{\hbar c} \Lambda(t_i)}, \tag{51}$$

the path integral expression remains true:

$$\langle \vec{x}_f, t_f |' \vec{x}_i, t_i \rangle' = \int \mathcal{D}\vec{x}(t) \exp \frac{i}{\hbar} \int_{t_i}^{t_f} dt \left(\frac{m}{2} \dot{\vec{x}}^2 - e\phi' + \frac{e}{c} \vec{A}' \cdot \dot{\vec{x}} \right). \tag{52}$$

Note that changing phases of states does not affect probabilities.

Because of this change in the position eigenstates, the wave functions are also changed correspondingly

$$\psi(\vec{x}, t) = \langle \vec{x}, t | \psi \rangle \to \psi'(\vec{x}, t) = \langle \vec{x}, t |' \psi \rangle = e^{-i \frac{e}{\hbar c} \Lambda(\vec{x}, t)} \psi(\vec{x}, t). \tag{53}$$

You can readily verify that the Schrödinger equation still takes the same form

$$i\hbar \frac{\partial}{\partial t} \psi' = \left[\frac{1}{2m} \left(\frac{\hbar}{i} \vec{\nabla} - \frac{e}{c} \vec{A}' \right)^2 + e\phi' \right] \psi' \tag{54}$$

in terms of new potentials and new wave function.

2.3 Aharonov–Bohm Effect

This subject is well covered in the book. The only additional discussion I have is the experimental work A. Tonomura, *et al*, "Evidence for Aharonov-Bohm effect with magnetic field completely shielded from electron wave," *Phys. Rev. Lett.* **56**, 792 (1986), done after Sakurai's death and hence not referred to in the book. This experiment uses electron holography to study the phase information of the electron wave function. One of the main criticisms on the experimental studies of Aharonov-Bohm effect had been that "well, how do you know that there *really* isn't any magnetic field leaked out to the region of electron propagation?" This work eliminated this concern completely, by wrapping the magnetic field with superconductor. Meißner effect in superconductivity guarantess that the magnetic field cannot leak out from the toroid they used. This work is widely considered as the definitive experiment.[2]

The scalar Aharonov–Bohm effect was verified only quite recently in neutron interferometry. See W.-T. Lee, O. Motrunich, B. E. Allman, and S. A. Werner, "Observation of Scalar Aharonov-Bohm Effect with Longitudinally Polarized Neutrons," *Phys. Rev. Lett.* **80**, 3165-3168 (1998).

2.4 Magnetic Monopole

If there exist a particle with a magenetic charge, with north- or south-pole only and not both, even just one in the entire Universe, one finds an interesting phenomenon: the quantum theory requires the electric charges to be quantized. This point was found by Dirac.

First of all, it is easy to add magnetic monopoles to the Maxwell's equations

$$\vec{\nabla} \cdot \vec{E} = 4\pi\rho, \tag{55}$$

$$\vec{\nabla} \times \vec{B} = \frac{1}{c}\dot{\vec{E}} + \frac{4\pi}{c}\vec{\jmath}, \tag{56}$$

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\dot{\vec{B}} - \frac{4\pi}{c}\vec{\jmath}_M, \tag{57}$$

$$\vec{\nabla} \cdot \vec{B} = 4\pi\rho_M. \tag{58}$$

Both the electric and magnetic charges are conserved,

$$\dot{\rho} + \vec{\nabla} \cdot \vec{\jmath} = 0, \tag{59}$$

$$\dot{\rho}_M + \vec{\nabla} \cdot \vec{\jmath}_M = 0. \tag{60}$$

[2] Actually, Nature was kind to us. When you wrap a magnetic flux with a superconductor, the amount of the magnetic flux is only allowed to be an integer multiple of $hc/2|e|$, not $hc/|e|$. This is because of the fact that a superconductor is a coherent phenomenon due to a Bose–Einstein condensate of Cooper pairs of charge $2e$. We will discuss it briefly in 221B. Thanks to this factor of two, the experiment could exhibit a phase shift of π, confirming Aharonov–Bohm effect dramatically.

At the classical level (namely with Maxwell's equations and the equation of motion for the point particle), there is no restriction on the size of electric and magnetic charges.

Sakurai presents the argument following the original one by Dirac. Here I present an alternative argument following a paper by Ed Witten, *Nucl. Phys.* **B223**, 422-432 (1983). Suppose the magnetic monopole is sitting at the origin, and a charged particle is moving around it, say on a sphere of radius r. Consider the path integral of the charged particle. Consider the amplitude $\langle \vec{x}, t_f | \vec{x}, t_i \rangle$ where the initial and the final points are the same. Note that we are not summing over $\vec{x}$ unlike in the partition function. In the path integral, the integrand for each path must be well-defined. It sounds like a trivial point, but it turns out it isn't. Look at the factor

$$e^{-\frac{i}{\hbar}\frac{e}{c}\oint \vec{A}\cdot d\vec{x}}. \tag{61}$$

Because we took the initial and final points the same, the exponent is a loop integral, and we can use Stokes' theorem to rewrite it as a surface integral

$$\oint \vec{A} \cdot d\vec{x} = \int \vec{B} \cdot d\vec{S}. \tag{62}$$

Here, $d\vec{S}$ is a surface element along its normal vector. The poins is that there are *two* ways to choose the surface to integrate over, on each side of the monopole. The difference between two ways is the entire surface integral,

$$\int \vec{B} \cdot d\vec{S} = 4\pi e_M, \tag{63}$$

where e_M is the magnetic charge of the monopole.

In order for the integrand of the path integral to be well-defined, the difference in the action shouldn't make any difference in the integrand,

$$e^{-\frac{i}{\hbar}\frac{e}{c}\int \vec{B}\cdot d\vec{S}} = e^{-i4\pi\frac{ee_M}{\hbar c}} = 1. \tag{64}$$

Therefore,

$$4\pi\frac{ee_M}{\hbar c} = 2\pi N, \qquad N \in \mathbb{Z}, \tag{65}$$

or equivalently,

$$\frac{2ee_M}{\hbar c} = N, \tag{66}$$

Eq. (2.6.85) of Sakurai.

We do not know if a magnetic monopole exists. However, the quantization of electric charges has been one of the puzzles in particle physics. We take it for granted that the hydrogen atom is electrically neutral. But why? The proton is made up of three quarks, two up-quarks of charge $\frac{2}{3}|e|$ and one down-quark of charge $-\frac{1}{3}|e|$. The total charge of the proton is $|e|$, which cancels the charge of the electron $e = -|e|$. But why do the electron and quarks have the charges they do? There is no satisfactory answer to this puzzle within the Standard Model of particle physics. If there is a reason why the electric charges are quantized in the unit of $\frac{1}{3}|e|$, it goes a long way towards a solution. That is why the existence of magnetic monopole is of our strong interest. There are extensions of the Standard Model which do explain this puzzle, such as grand-unified theories. In these theories, it turns out, magnetic monopoles are indeed predicted. The search for magnetic monopoles are still continuing.[3]

3 Rotating Frame

Physics in a rotating frame has nothing to do with the electromagnetism, but there is a strong similarity in the formalism.

3.1 Classical Mechanics

Particles in inertial frames have the ordinary Lagrangian

$$L = \frac{m}{2}\dot{\vec{x}}^2 - V. \tag{67}$$

Here, the position $\vec{x}$ is defined relative to the center of the Earth. On the surface of the Earth, however, it is much more convenient to use the coordinate system that rotates together with the Earth. (We are ignoring the efffects of Earth's revolution around the Sun, revolution of the solar system inside the Milky Way, and the infall of the Milky Way towards the Virgo cluster, entirely.) Suppose we take the axis of rotation to be the z-axis, pointing from the South to the North pole, and other orthogonal directions x and y. The equator is then entirely on the xy plane. The coordinate system fixed on the

[3]There was once a claim that a magnetic monopole was found experimentally. This result had not been confirmed by other experiments. It is still logically possible that he detected the only monopole in our Universe!

Earth $\vec{x}'$ rotates relative to the inertial frame as the Earth rotates around the z-axis,

$$\vec{x}' = (x', y', z') = (x \cos \omega t - y \sin \omega t, x \sin \omega t + y \cos \omega t, z). \tag{68}$$

Solving is backwards, we can express the inertial coordinates $\vec{x}$ in terms of those fixed on the Earth,

$$\vec{x} = (x, y, z) = (x' \cos \omega t + y' \sin \omega t, x' \sin \omega t - y' \cos \omega t, z'). \tag{69}$$

We can now substitute these expressions into the Lagrangian. First,

$$\begin{aligned} \dot{\vec{x}} &= (\dot{x}' \cos \omega t + \dot{y}' \sin \omega t, \dot{x}' \sin \omega t - \dot{y}' \cos \omega t, \dot{z}') \\ &\quad + \omega(-x' \sin \omega t + y' \cos \omega t, x' \cos \omega t + y' \sin \omega t, 0) \\ &= ((\dot{x}' + \omega y') \cos \omega t + (\dot{y}' - \omega x') \sin \omega t, \\ &\quad (\dot{x}' + \omega y') \sin \omega t - (\dot{y}' - \omega x') \cos \omega t, \dot{z}'). \end{aligned} \tag{70}$$

We then find the Lagrangian

$$L = \frac{m}{2}((\dot{x}' + \omega y)^2 + (\dot{y}' - \omega x')^2 + \dot{z}'^2) - V. \tag{71}$$

To simplify the notation, we define the angular velocity vector $\vec{\omega} = (0, 0, \omega)$, such that the Lagrangian is

$$L = \frac{m}{2}(\dot{\vec{x}}' - \vec{\omega} \times \vec{x}')^2 - V. \tag{72}$$

With this form, we are no longer required to take the Earth's rotation axis as the z-axis of our coordinate system.

The Euler–Lagrange equation from this Lagrangian is

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\vec{x}}'} - \frac{\partial L}{\partial \vec{x}'} = m\ddot{\vec{x}}' - m\vec{\omega} \times \dot{\vec{x}}' + \frac{\partial V}{\partial \vec{x}'} = 0. \tag{73}$$

The term $m\vec{\omega} \times \dot{\vec{x}}'$ represents the Coriolis force.

Looking at the Lagrangian Eq. (72), and expanding the square, we can identify a "vector potential"

$$\frac{e}{c}\vec{A} = -m\vec{\omega} \times \vec{x}' \tag{74}$$

and a “scalar potential”

$$e\phi = V - \frac{m}{2}(\omega \times \vec{x}')^2. \tag{75}$$

Therefore, we can use the same formalism we used with the electromagnetic couplings in this problem.

Given the identification of a “vector potential” and a “scalar potential,” we can write down the Hamiltonian immediately by inspection,

$$H = \frac{1}{2m}\left(\vec{p}' + m\vec{\omega} \times \vec{x}'\right)^2 + V - \frac{m}{2}(\omega \times \vec{x}')^2. \tag{76}$$

Here, $\vec{p}'$ is the canonical momentum conjugate to the rotating coordinate $\vec{x}'$.

3.2 Quantum Mechanics

In quantum mechanics, we can expect “Aharonov–Bohm effect” of the “vector potential” in the rotating frame. In this case, it is not that there is no “magnetic field.” There is. But it is not necessary to have the “magnetic field” vanish in order to have the Aharonov–Bohm phase. Recalling the derivation of the AB effect, you always find there must be a relative phase between two waves given by the amount of the “magnetic flux” between two paths. Therefore, we do expect the “Aharonov–Bohm effect” in the rotating frame. Of course, once there is a “magnetic field,” there is a “Lorentz force,” which is nothing but the Coriolis force in this case. But we can well have the situation where the deflection of the particle due to Coriolis force is negligible while the phase difference can be sizable. One can also make sure that the difference due to the gravitational potential and the additional “scalar potential” cancels between two paths to single out the “Aharonov–Bohm phase”.

The phase difference between two paths is given by the “vector potential” Eq. (74),

$$\exp i\frac{e}{\hbar c}\oint \vec{A}\cdot d\vec{x}' = \exp i\frac{1}{\hbar}\oint (-m\vec{\omega} \times \vec{x}')\cdot d\vec{x}'. \tag{77}$$

Even though the position vector is defined always relative to the center of the Earth, we do not need to go all the way to the center of the Earth to define this integral. If you decompose $\vec{x}' = \vec{x}_0 + \vec{a}$, where $\vec{x}_0$ is the center of the interferometer, the contribution from $\vec{x}_0$ vanishes upon the loop integral. The only piece that counts is the remainder $\vec{a}$. Then rewriting $(\vec{\omega} \times \vec{a})\cdot d\vec{x}' =$

$\vec{\omega} \cdot (\vec{a} \times d\vec{x}')$, the combination in the parentheses gives (twice) the area the vector $\vec{a}$ sweeps, normal to $\vec{\omega}$ direction. Therefore, the phase difference is given purely geometrically as

$$\exp i \frac{-2m}{\hbar} \vec{\omega} \cdot \vec{A}, \tag{78}$$

where $\vec{A}$ is a vector whose length is the area and direction is along its normal. Another way to derive the same result is to note that the "magnetic field" is

$$\frac{e}{c} \vec{B} = \vec{\nabla} \times (-m\vec{\omega} \times \vec{x}') = -2m\vec{\omega}. \tag{79}$$

Then the phase factor is nothing but due to the "magnetic flux" going between two paths.

The paper S. A. Werner, J. -L. Staudenmann, and R. Colella, "Effect of Earth's Rotation on the Quantum Mechanical Phase of the Neutron," *Phys. Rev. Lett.* **42**, 1103 (1979), claimed to have detected this effect, by observing the variation in counting rate as they moved the area of the interferometer relative to the Earth's rotational axis while making sure that difference in the gravitational and the "scalar potential" cancel between two paths. I find this result stunning.