■ HW #4 (221b)

1)

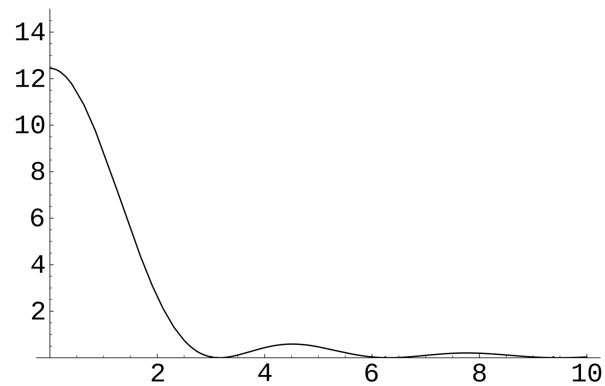
a) Set $\hbar$=m=a=1.

```
δ0[k_, γ_] := 1/(2 I) Log[(1 + (2γ/k) E^(-I k) Sin[k]) / (1 + (2γ/k) E^(I k) Sin[k])]

σ0[k_, γ_] := (4 π)/k^2 Sin[δ0[k, γ]]^2
```

(γ=100): Large γ emulates hard sphere scattering except at resonances. The graph below essentially matches that of the hardsphere cross section $\frac{4\pi}{k^2}\sin^2 ka$; in particular, $\sigma_0 \to 4\pi$ as $k \to 0$.

```
Plot[σ0[k, 100], {k, 0, 10}, PlotRange → {0, 15}]
```



When we zoom in (γ=100 still), we see the expected resonances at $k\,a \approx n\,\pi$, corresponding to the almost−bound states inside the shell.

```
Plot[σ0[k, 100], {k, 2, 10}, PlotRange → {0, 2}]
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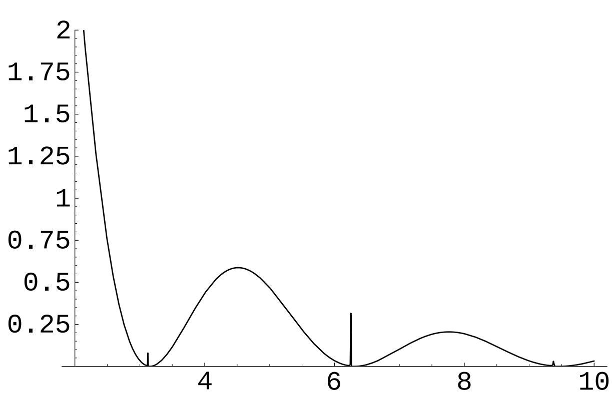
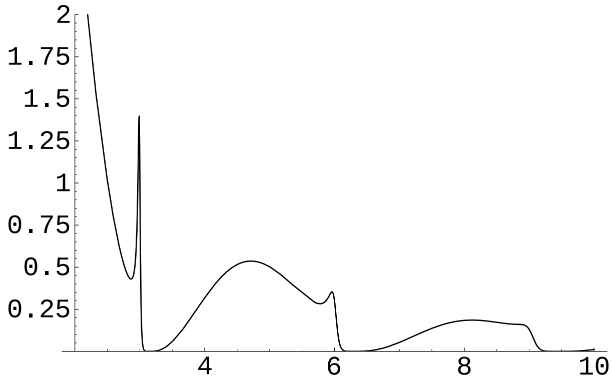


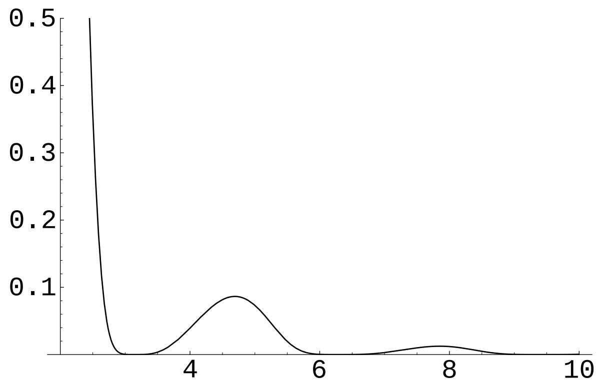
(γ=10): For middling γ, the resonances are shifted towards smaller k values according to equation (III.54) of the lecture notes. The peakes are higher and wider than the γ=100 resonances because of the increased width Γ. (See equation III.63, which is valid for poles close to the real axis.)

```
Plot[σ0[k, 10], {k, 2, 10}, PlotRange → {0, 2}]
```



(γ=1): For small γ, we can't distinguish the resonances from the oscillatory background. By III.52, small γ means poles far from the real axis, so the disappearance of the resonances confirms that the particle doesn't 'feel' poles far from its real momentum.

```
Plot[σ0[k, 1], {k, 2, 10}, PlotRange → {0, .5}]
```



b) Lecture notes 'Scattering Theory III.'

c) The wavefunctions are derived in the lecture notes III. Following the conventions there but setting $\hbar$=m=a=1, $k = k_0 - i\,\kappa$, $E = E_0 - i\,\Gamma/2 = k_0{}^2/2 - i\,k_0\,\kappa + O(\kappa^2)$. I choose the approximate value of k at the second resonance and leave γ arbitrary for now.

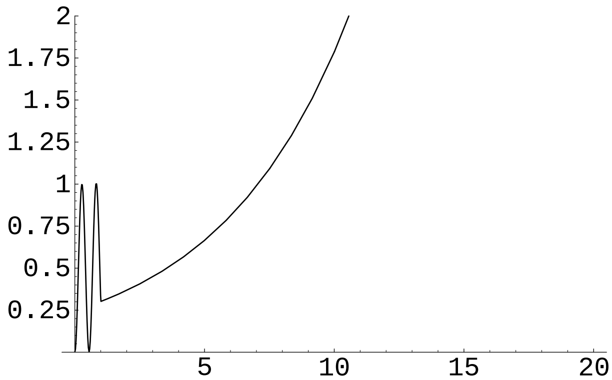
$$E_0 = k0^2;\ \Gamma = 2\,k0\,\kappa;\ k0 = \frac{2\pi}{1+\frac{1}{2\gamma}};\ \kappa = \left(\frac{2\pi}{4\gamma}\right)^2;$$

The wavefunction, defined piecewise on the intervals r=(0,1), r=(1,∞), is

```
rR[r_, t_] :=
  If[r < 1, Sin[(k0 - I κ) r] E^(-I E0 t) E^(-Γ t/2), Sin[(k0 - I κ)] E^(I (k0-I κ) (r-1)) E^(-I E0 t) E^(-Γ t/2)];
```

Double click on the graph to watch the animation (γ=5):

```
Do[Plot[Abs[rR[r, i]]^2 /. {γ → 5, t → 2 i}, {r, 0, 20}, PlotRange → {0, 2}], {i, 0, 20}]
```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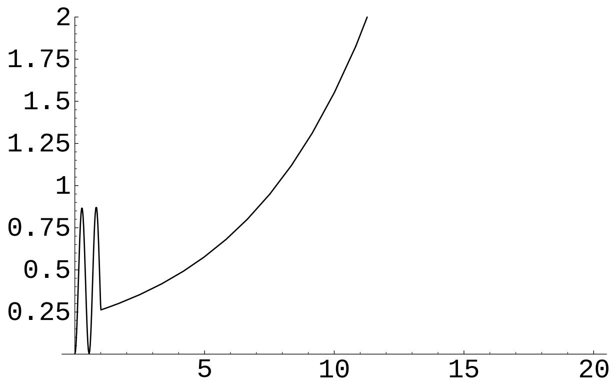
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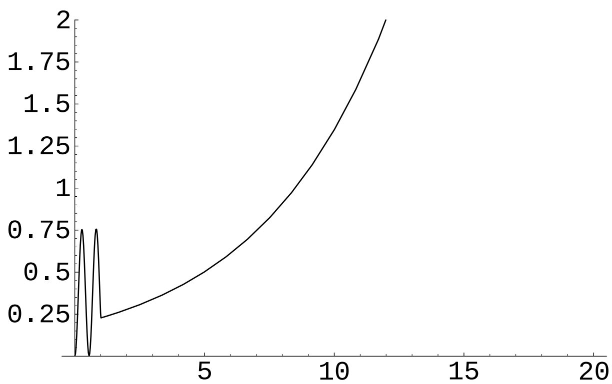
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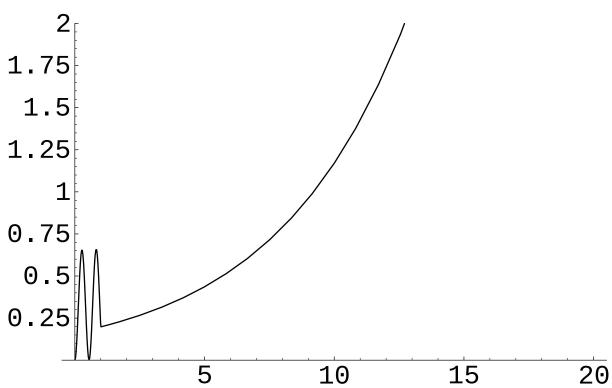
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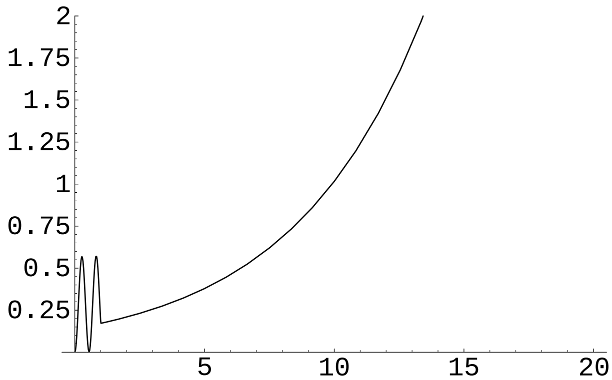
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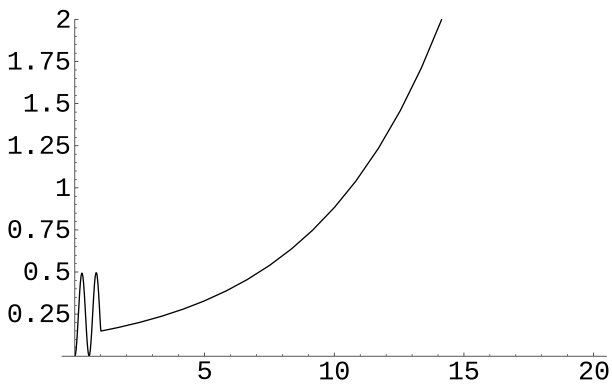
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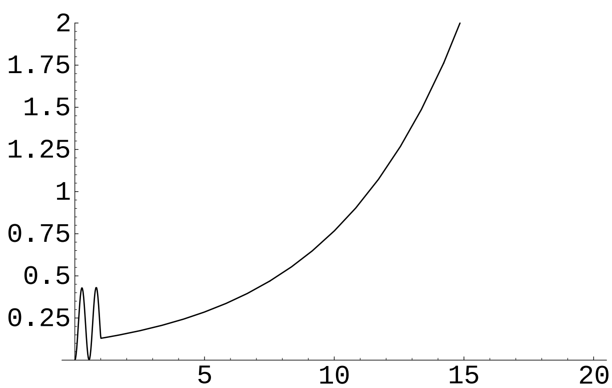
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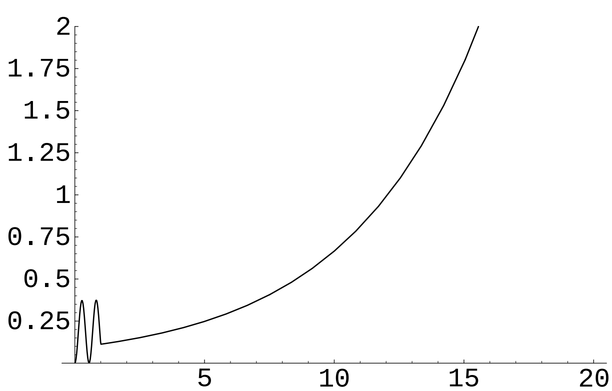
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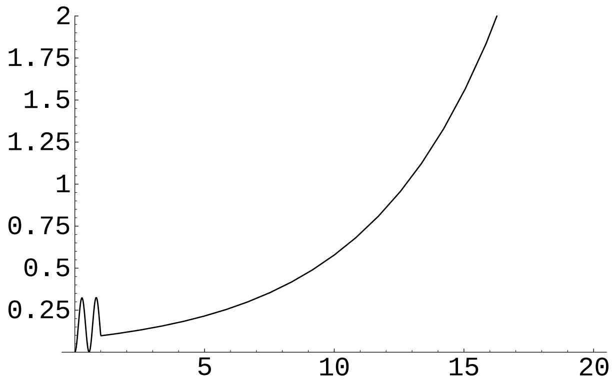
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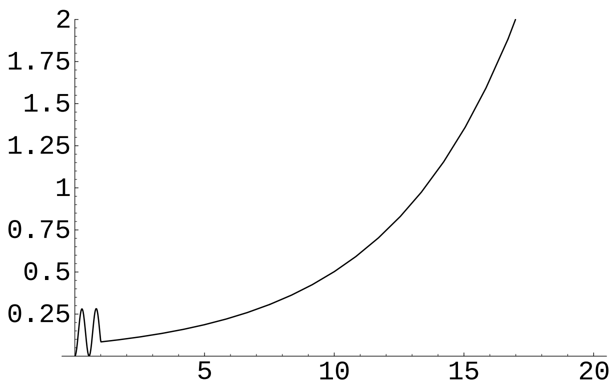
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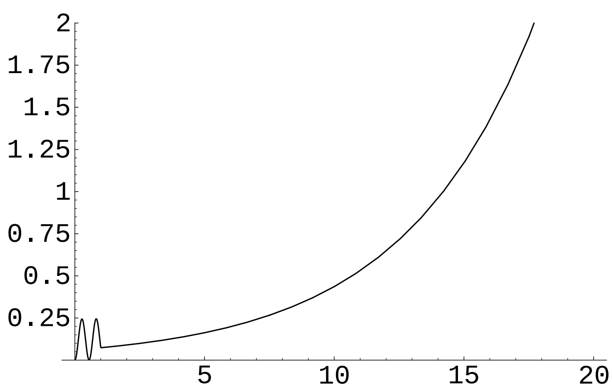
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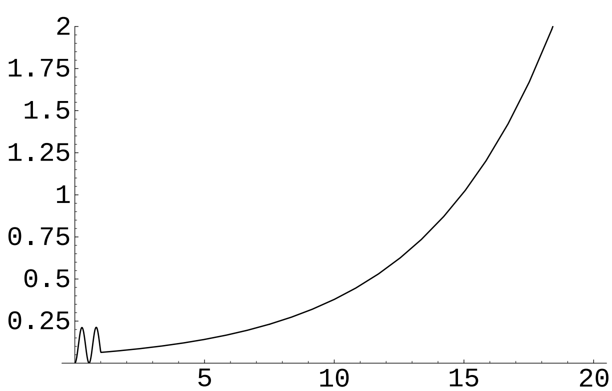
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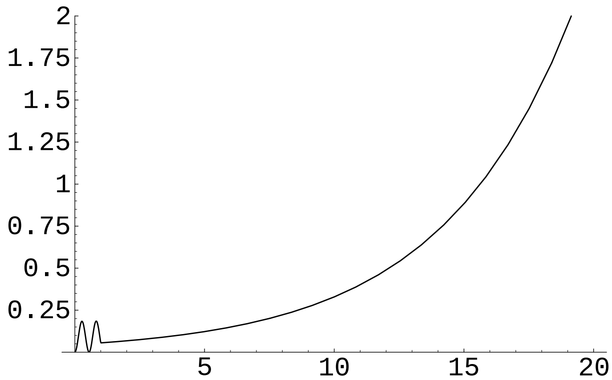
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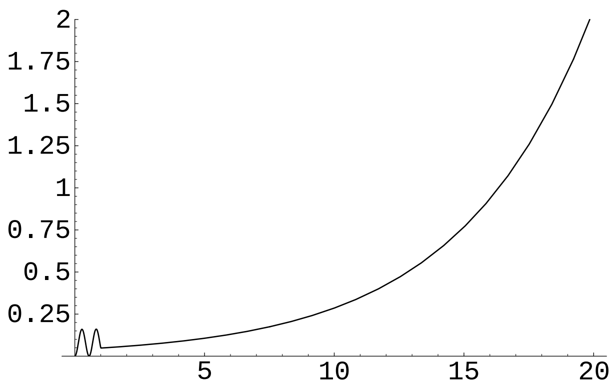
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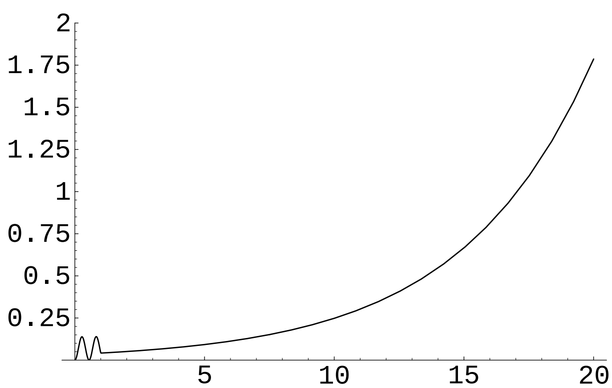
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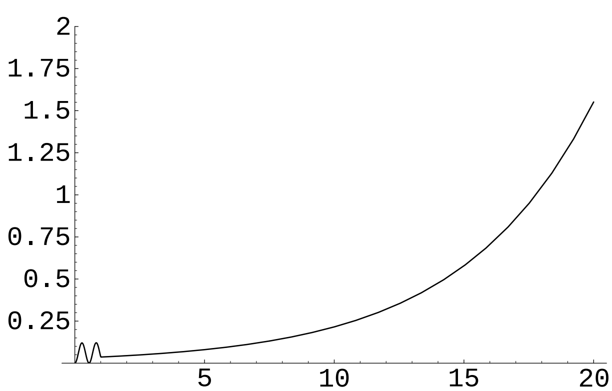
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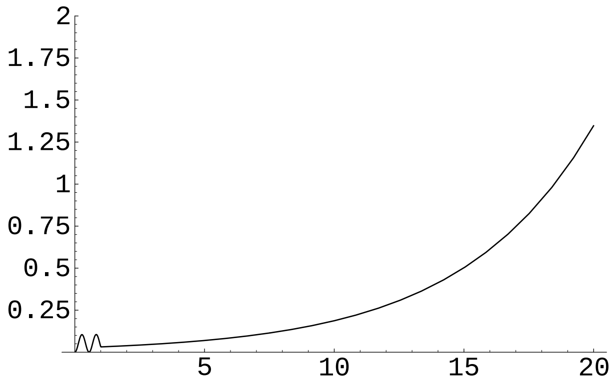
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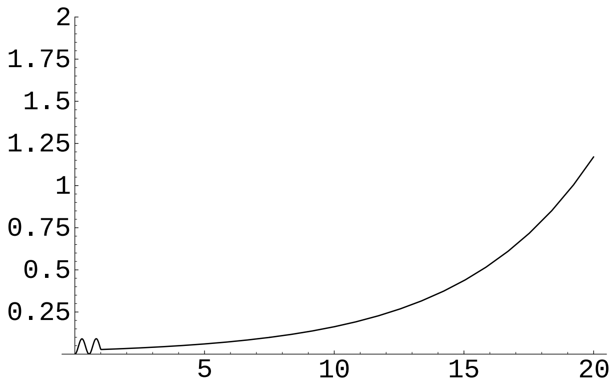
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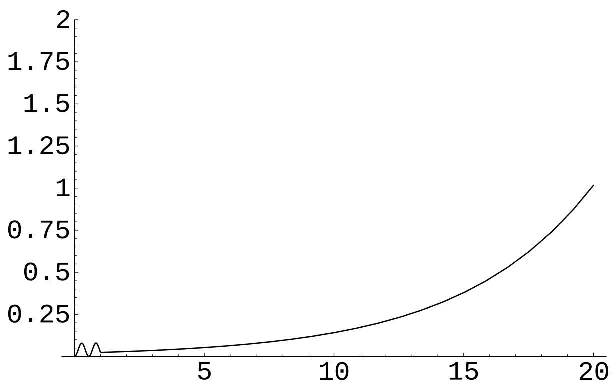
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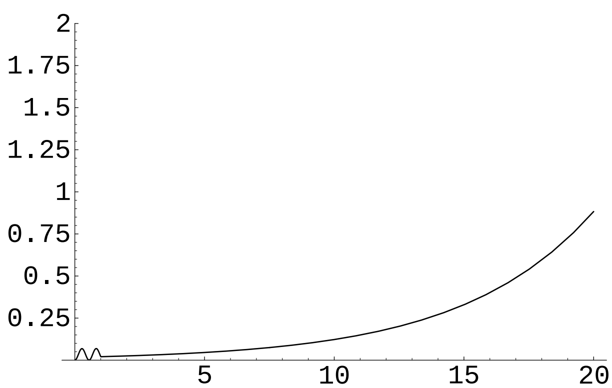
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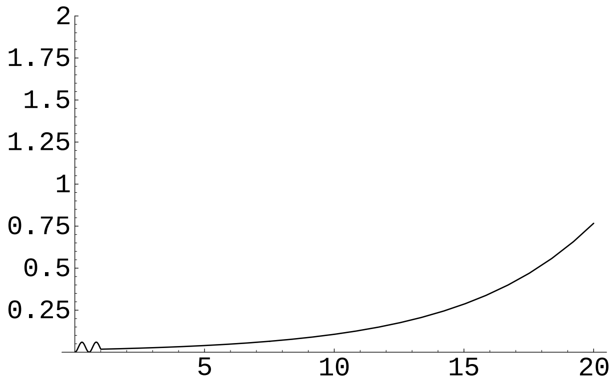

We can show analytically that the decrease in probability intside the shell is equal to the probability flux flowing out through the shell. With $\rho = |\psi|^2$, $j = \frac{\hbar}{2mi}(\psi^* \nabla\psi - \psi\nabla\psi^*)$, we want to show $\int \frac{d\rho}{dt}\, dV + \int j\cdot da = 0$. In particular, we integrate $\frac{d\rho}{dt}$ over the volume inside r < a integrate *j* over the sphere at r = a. For the *j* integral, we can use the expression for the wavefunction just inside or just outside r = a. The problem asks that we use the latter.

Using the above wavefunction with all constants restored,

$$\frac{d\rho}{dt} = \frac{-\Gamma}{\hbar}\rho = \frac{-\Gamma}{2\hbar r^2}(\cosh 2\kappa r - \cos 2k_0 r)\, e^{-\Gamma t/\hbar},$$

so that

$$\int \frac{d\rho}{dt}\, dV = \int_0^a \frac{d\rho}{dt}\, 4\pi r^2 = \frac{-2\pi\Gamma}{\hbar}\left(\frac{\sinh 2\kappa a}{2\kappa} - \frac{\sin 2k_0 a}{2k_0}\right) e^{-\Gamma t/\hbar}.$$

A similar computation for *j* using the r > a wavefunction gives

$$j = \frac{\hbar}{2mi}(ik\hat{r} + ik^*\hat{r})\rho = \frac{\hbar k_0}{2m}\frac{\hat{r}}{r^2}(\cosh 2\kappa r - \cos 2k_0 r)\, e^{-\Gamma t/\hbar},$$

and

$$\int_{r=a} j\cdot da = 4\pi a^2 j = \frac{2\pi\hbar k_0}{m}(\cosh 2\kappa a - \cos 2k_0 a)\, e^{-\Gamma t/\hbar}.$$

Plugging in $\Gamma = \frac{2\hbar Ò k_0 \kappa}{m}$,

$$\int \frac{d\rho}{dt}\, dV + \int j\cdot da = 0$$
$$\Longleftrightarrow$$
$$-\frac{4\pi\hbar^2 k_0 \kappa}{m}\left(\frac{\sinh 2\kappa a}{2\kappa} - \frac{\sin 2k_0 a}{2k_0}\right) e^{-\Gamma t/\hbar} + \frac{2\pi\hbar k_0}{m}(\cosh 2\kappa a - \cos 2k_0 a)\, e^{-\Gamma t/\hbar} = 0$$
$$\Longleftrightarrow$$
$$\sinh 2\kappa a - \frac{\kappa}{k_0}\sin 2k_0 a = \cosh 2\kappa a - \cos 2k_0 a.$$

To demonstrate this last equality, we must use the condition we developed when finding this solution to the Schrodinger equation, i.e. the condition from matching the discontinuity in derivatives at r = a with the delta function singularity. Equivalently, this is the condition for having a pole in the scattering amplitude: $e^{Òika} = 1 - 2i\frac{\hbar^2 k}{2m\gamma}$. Plugging in $k = k_0 - i\kappa$, the real and imaginary parts of this equation read

$$e^{2\kappa a}\cos 2k_0 a = 1 - \frac{\hbar^2 \kappa}{m\gamma} \quad \text{and} \quad e^{2\kappa a}\sin 2k_0 a = -\frac{\hbar^2 k_0}{m\gamma}.$$

Plugging these expressions into our condition

$$\sinh 2\kappa a - \frac{\kappa}{k_0} \sin 2 k_0 a = \cosh 2\kappa a - \cos 2 k_0 a$$

gives

$$\sinh 2\kappa a + \frac{\hbar^2 \kappa}{m\gamma} e^{-2\kappa a} = \cosh 2\kappa a - \left(1 - \frac{\hbar^2 \kappa}{m\gamma}\right) e^{-2\kappa a}$$

or

$$\sinh 2\kappa a = \cosh 2\kappa a - e^{-2\kappa a},$$

which does in fact hold.