■ WKB approximation to the harmonic oscillator wave functions

■ We take m=1, ω=1, $\hbar$=1 and work out WKB wave function

■ First, classically allowed region

In[1]:= $\text{Integrate}\left[\sqrt{(2n+1)-Q^2}, Q\right]$

Out[1]= $\frac{1}{2} Q \sqrt{1+2n-Q^2} - \frac{1}{2}(1+2n)\,\text{ArcTan}\left[\frac{Q\sqrt{1+2n-Q^2}}{-1-2n+Q^2}\right]$

In[2]:= $\text{Limit}\left[\%, Q \to -\sqrt{2n+1}\right]$

Out[2]= $-\frac{1}{4}(1+2n)\pi$

In[3]:= $u = \frac{1}{(2n+1-Q^2)^{1/4}} \text{Cos}\left[\%\% - \% - \frac{\pi}{4}\right]$

Out[3]= $\frac{1}{(1+2n-Q^2)^{1/4}}\left(\text{Cos}\left[\frac{\pi}{4} - \frac{1}{4}(1+2n)\pi - \frac{1}{2}Q\sqrt{1+2n-Q^2} + \frac{1}{2}(1+2n)\,\text{ArcTan}\left[\frac{Q\sqrt{1+2n-Q^2}}{-1-2n+Q^2}\right]\right]\right)$

■ Next, classically forbidden region

In[4]:= $\text{Integrate}\left[\sqrt{-(2n+1)+Q^2}, Q\right]$

Out[4]= $\frac{1}{2} Q\sqrt{-1-2n+Q^2} + \frac{1}{2}(-1-2n)\,\text{Log}\left[Q+\sqrt{-1-2n+Q^2}\right]$

In[5]:= $\text{Limit}\left[\%, Q \to -\sqrt{2n+1}\right]$

Out[5]= $-\frac{1}{2}(1+2n)\,\text{Log}\left[-\sqrt{1+2n}\right]$

In[6]:= $v1 = \frac{1}{2}\frac{1}{(-(2n+1)+Q^2)^{1/4}} \text{Exp}[-\% + \%\%]$

Out[6]= $\frac{E^{\frac{1}{2}Q\sqrt{-1-2n+Q^2}+\frac{1}{2}(1+2n)\,\text{Log}\left[-\sqrt{1+2n}\right]+\frac{1}{2}(-1-2n)\,\text{Log}\left[Q+\sqrt{-1-2n+Q^2}\right]}}{2(-1-2n+Q^2)^{1/4}}$

In[7]:= $\text{Integrate}\left[\sqrt{-(2n+1)+Q^2}, Q\right]$

Out[7]= $\frac{1}{2} Q\sqrt{-1-2n+Q^2} + \frac{1}{2}(-1-2n)\,\text{Log}\left[Q+\sqrt{-1-2n+Q^2}\right]$

In[8]:= $\text{Limit}\left[\%, Q \to \sqrt{2n+1}\right]$

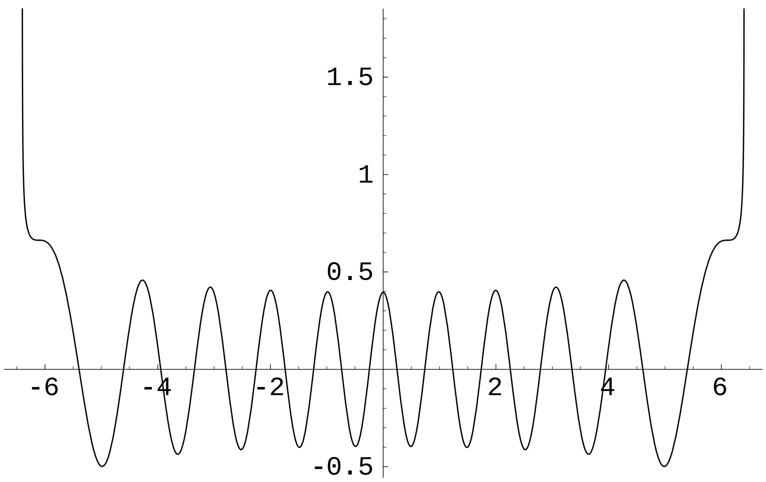
Out[8]= $-\frac{1}{4}(1+2n)\,\text{Log}[1+2n]$

In[9]:= $\mathbf{v2 = \frac{1}{2}\,\frac{1}{(-(2\,n+1)+Q^2)^{1/4}}\,Exp[\%-\%\%]}$

Out[9]= $\frac{E^{-\frac{1}{2}Q\sqrt{-1-2n+Q^2}-\frac{1}{4}(1+2n)\,Log[1+2n]-\frac{1}{2}(-1-2n)\,Log\left[Q+\sqrt{-1-2n+Q^2}\right]}}{2\,(-1-2n+Q^2)^{1/4}}$

- **n=20**

In[10]:= $\mathbf{uplot = Plot\left[u\ /.\ \{n\to 20\},\ \left\{Q,\ -\sqrt{41},\ \sqrt{41}\right\},\ PlotPoints\to 200\right]}$



Out[10]= - Graphics -

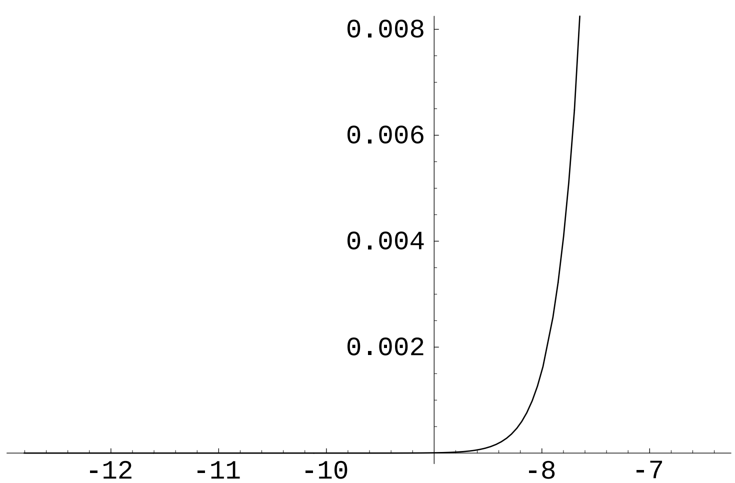
In[11]:= $\mathbf{v1plot = Plot\left[v1\ /.\ \{n\to 20\},\ \left\{Q,\ -2\sqrt{41},\ \sqrt{41}\right\},\ PlotPoints\to 200\right]}$

Plot::plnr : v1 /. {n → 20} is not a machine-size real number at Q = -6.34671.

Plot::plnr : v1 /. {n → 20} is not a machine-size real number at Q = -6.39185.

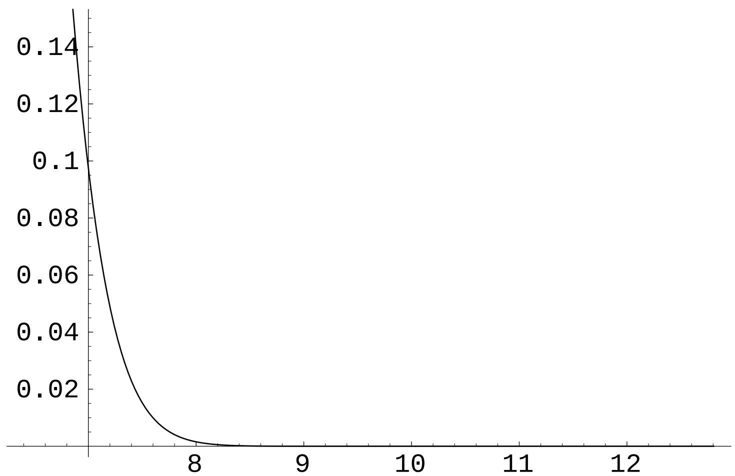
Plot::plnr : v1 /. {n → 20} is not a machine-size real number at Q = -6.39787.

General::stop : Further output of Plot::plnr will be suppressed during this calculation.



Out[11]= - Graphics -

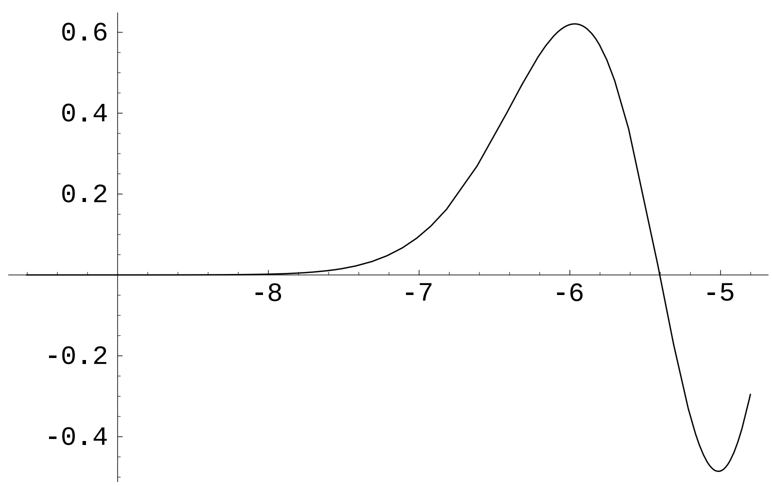
In[12]:= **v2plot = Plot[v2 /. {n → 20}, {Q, $\sqrt{41}$, 2 $\sqrt{41}$}, PlotPoints → 200]**



Out[12]= - Graphics -

In[13]:= **a1plot = Plot[$\left(\frac{\pi}{(2\sqrt{2n+1})^{1/3}}\right)^{1/2}$ AiryAi[$(2\sqrt{2n+1})^{1/3}$ $(-Q-\sqrt{2n+1})$] /. {n → 20}, {Q, $\frac{-3}{2}\sqrt{41}$, $\frac{-3}{4}\sqrt{41}$}]**

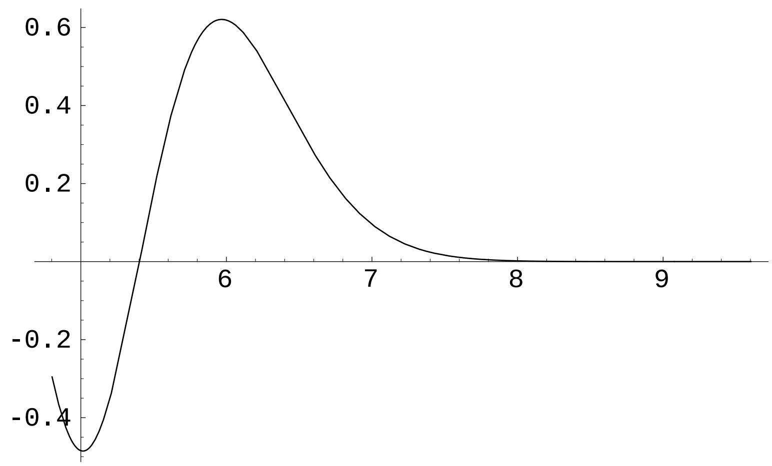
General::spell1 :
Possible spelling error: new symbol name "a1plot" is similar to existing symbol "v1plot".



Out[13]= - Graphics -

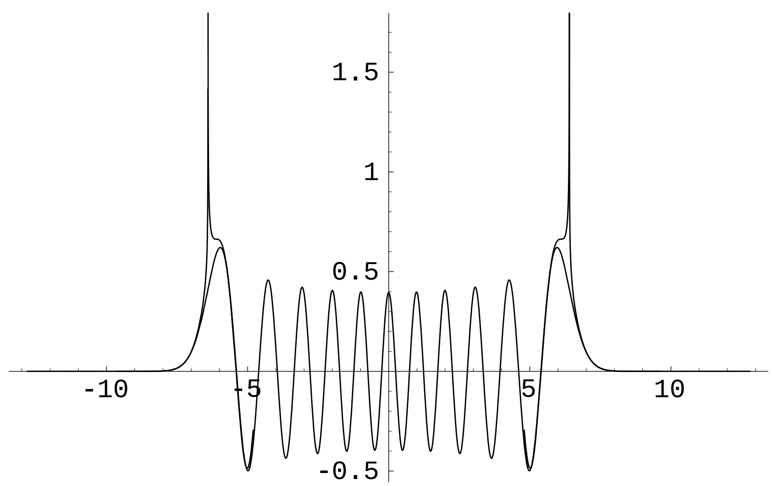
In[14]:= **a2plot = Plot[** $\left(\frac{\pi}{\left(2\sqrt{2n+1}\right)^{1/3}}\right)^{1/2}$ **AiryAi[** $\left(2\sqrt{2n+1}\right)^{1/3}\left(Q-\sqrt{2n+1}\right)$ **] /. {n → 20},**
{Q, $\frac{3}{4}\sqrt{41}$**,** $\frac{3}{2}\sqrt{41}$ **}]**

```
General::spell1 :
 Possible spelling error: new symbol name "a2plot" is similar to existing symbol "v2plot".
```



Out[14]= - Graphics -

In[15]:= **WKBplot = Show[uplot, v1plot, a1plot, v2plot, a2plot]**

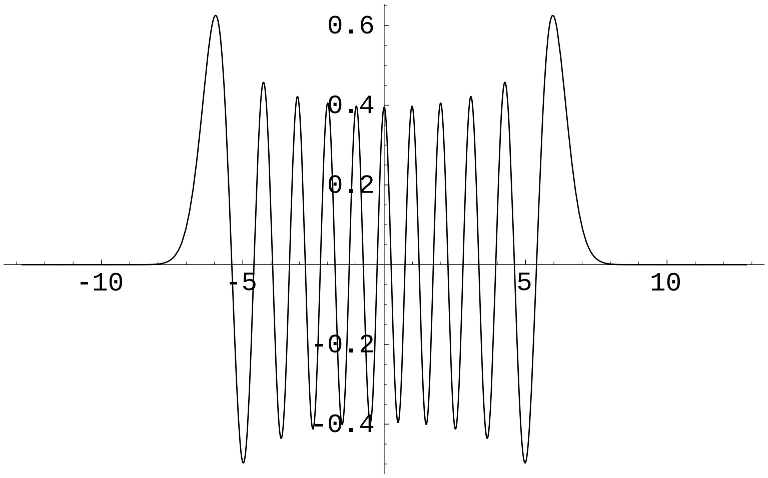


Out[15]= - Graphics -

In[16]:= **N[** $\frac{u}{\left(\sqrt{\pi}\,2^n\,n!\right)^{-1/2}\,\text{HermiteH}[n,Q]\,E^{-Q^2/2}}$ **/. {n → 20} /. {Q → 0}]**

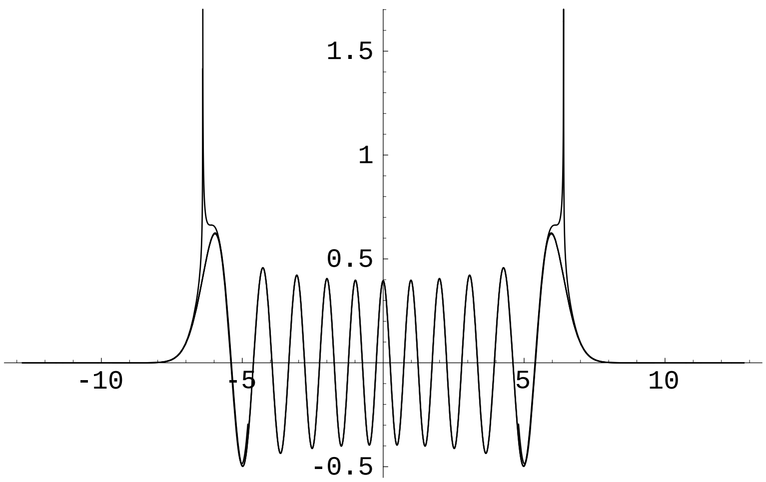
Out[16]= 1.25341

In[17]:= **trueplot = Plot**$\left[1.253407199896294\grave{}\ \left(\sqrt{\pi}\ 2^{n}\ n!\right)^{-1/2}\ \text{HermiteH}[n, x]\ E^{-x^{2}/2}\ /.\ \{n \to 20\},\ \left\{x, -2\sqrt{41}, 2\sqrt{41}\right\}, \text{PlotPoints} \to 200\right]$



Out[17]= - Graphics -

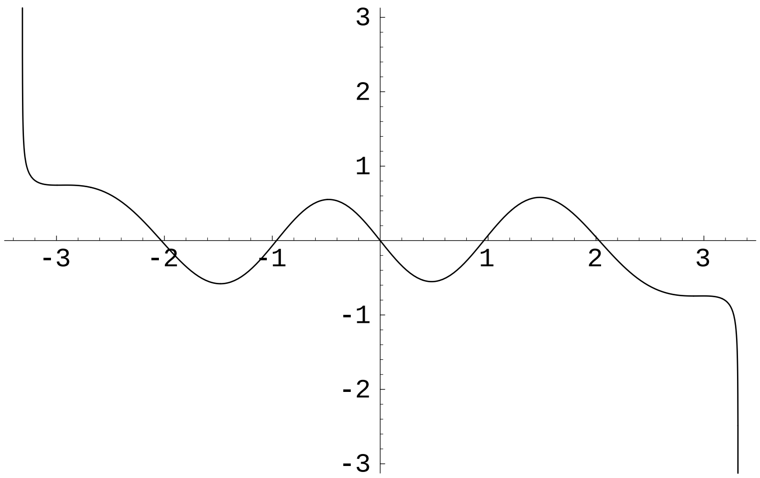
In[18]:= **Show[WKBplot, trueplot]**



Out[18]= - Graphics -

■ n=5

In[19]:= **uplot = Plot[u /. {n → 5}, {Q, -$\sqrt{11}$, $\sqrt{11}$}, PlotPoints → 200]**



Out[19]= - Graphics -

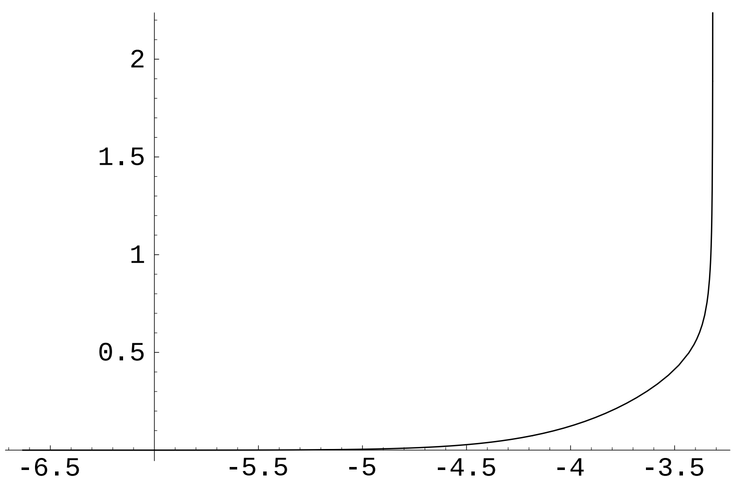
In[20]:= **v1plot = Plot[v1 /. {n → 5}, {Q, -2 $\sqrt{11}$, $\sqrt{11}$}, PlotPoints → 200]**

Plot::plnr : v1 /. {n → 5} is not a machine-size real number at Q = -3.28139.

Plot::plnr : v1 /. {n → 5} is not a machine-size real number at Q = -3.30781.

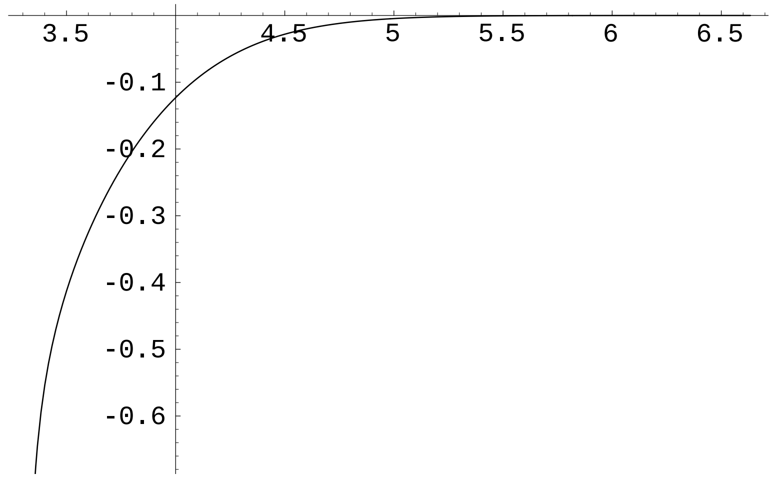
Plot::plnr : v1 /. {n → 5} is not a machine-size real number at Q = -3.3139.

General::stop : Further output of Plot::plnr will be suppressed during this calculation.



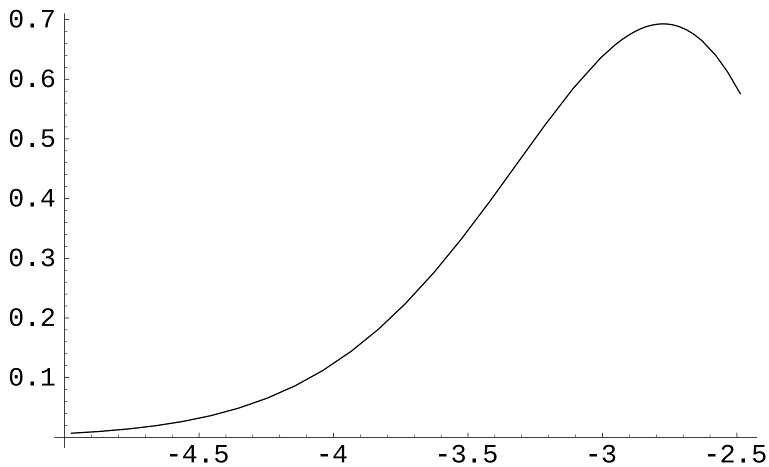
Out[20]= - Graphics -

In[21]:= **v2plot = Plot[-v2 /. {n → 5}, {Q, $\sqrt{11}$, 2 $\sqrt{11}$}, PlotPoints → 200]**



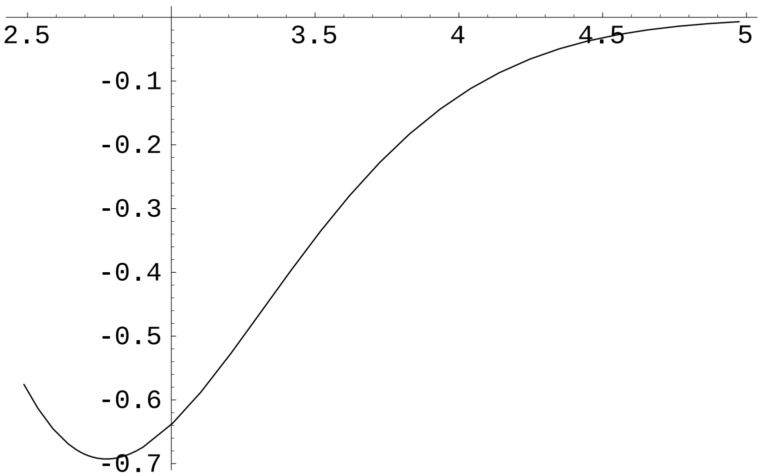
Out[21]= - Graphics -

In[22]:= **a1plot = Plot[$\left(\frac{\pi}{\left(2\sqrt{2n+1}\right)^{1/3}}\right)^{1/2}$ AiryAi[$\left(2\sqrt{2n+1}\right)^{1/3}\left(-Q-\sqrt{2n+1}\right)$] /. {n → 5}, {Q, $\frac{-3}{2}\sqrt{11}$, $\frac{-3}{4}\sqrt{11}$}]**



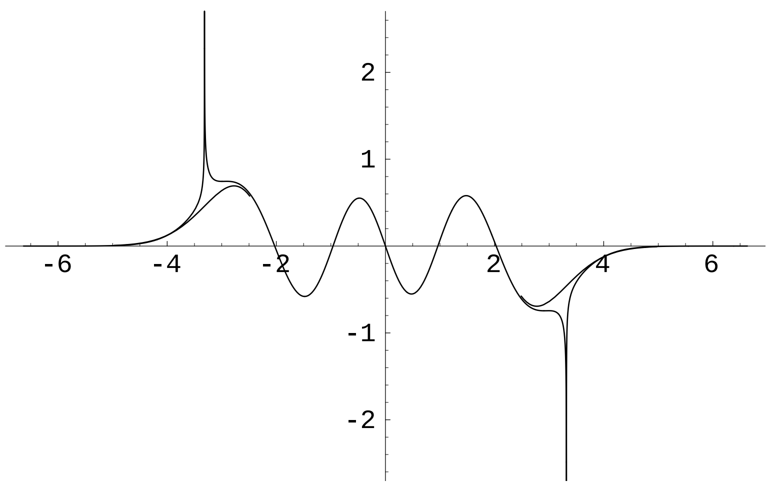
Out[22]= - Graphics -

In[23]:= **a2plot = Plot**$\left[-\left(\frac{\pi}{\left(2\sqrt{2n+1}\right)^{1/3}}\right)^{1/2} \text{AiryAi}\left[\left(2\sqrt{2n+1}\right)^{1/3}\left(Q-\sqrt{2n+1}\right)\right] /. \{n\to 5\}, \left\{Q, \frac{3}{4}\sqrt{11}, \frac{3}{2}\sqrt{11}\right\}\right]$



Out[23]= - Graphics -

In[24]:= **WKBplot = Show[uplot, v1plot, a1plot, v2plot, a2plot]**

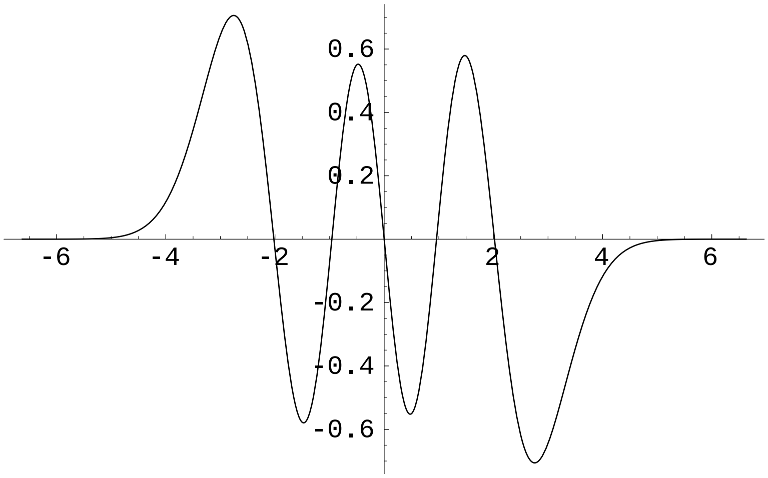


Out[24]= - Graphics -

In[25]:= $\text{N}\left[\frac{u}{\left(\sqrt{\pi}\,2^n\,n!\right)^{-1/2}\,\text{HermiteH}[n, Q]\,E^{-Q^2/2}} /. \{n\to 5\} /. \{Q\to 0.5\}\right]$

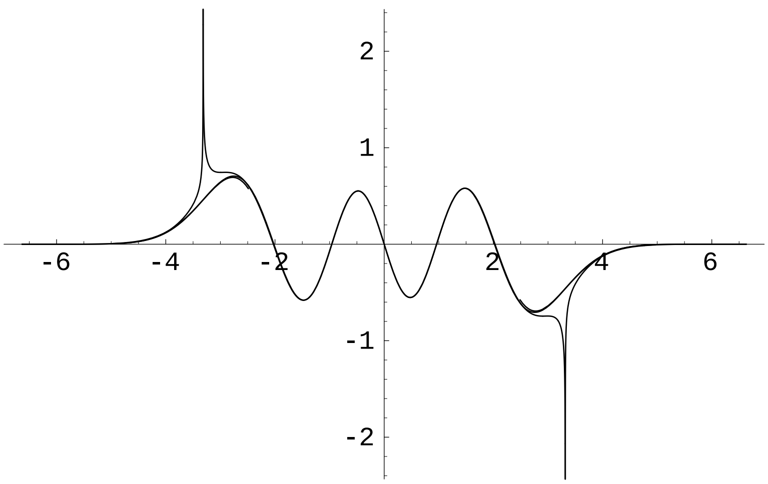
Out[25]= −1.25508

In[26]:= **trueplot = Plot[-1.2550763185968383` $\left(\sqrt{\pi}\ 2^n\ n!\right)^{-1/2}$ HermiteH[n, x] $E^{-x^2/2}$ /. {n → 5},
{x, -2 $\sqrt{11}$, 2 $\sqrt{11}$}, PlotPoints → 200]**



Out[26]= - Graphics -

In[27]:= **Show[WKBplot, trueplot]**



Out[27]= - Graphics -