221B Lecture Notes on Resonances in Classical Mechanics

1 Harmonic Oscillators

Harmonic oscillators appear in many different contexts in classical mechanics. Examples include: spring, pendulum (with a small amplitude approximation), electric circuit with a capacitor and a coil, antenna, a single harmonics of vibrating string or cavity or membrane, etc etc. The common equation to harmonic oscillators is the equation of motion

$$\ddot{x} + \omega_0^2 x = 0, \tag{1}$$

with the well-known solutions

$$x(t) = c_1 \cos\omega_0 t + c_2 \sin\omega_0 t. \tag{2}$$

2 Resonances Without Friction

We now exert an external force on a harmonic oscillator. We choose in particular a periodic force, $F = mf\cos\omega t$ (the factor of m is there only for the convenience). Then the equation of motion is

$$\ddot{x} + \omega_0^2 x = f\cos\omega t. \tag{3}$$

As is known in the theory of linear differential equations, any solution to an inhomogenous equation is given by a sum of a general solution to the homogenous equation and a solution to the inhomogeneous equation. The homogeneous solution is the same as the case without an external force, and an inhomogenous solution is easy to guess:[1]

$$x(t)_{inhom} = -\frac{f}{\omega^2 - \omega_0^2}\cos\omega t. \tag{4}$$

If we impose the initial condition $x(0) = 0$, $\dot{x}(0) = 0$, the solution is given by

$$x(t) = -\frac{f}{\omega^2 - \omega_0^2}(\cos\omega t - \cos\omega_0 t). \tag{5}$$

[1] You can also follow the derivation which we use in the case with friction below to obtain it without any guessing.

When $\omega \neq \omega_0$, two oscillatory terms randomly add or subtract. It is useful to rewrite the solution as

$$x(t) = -\frac{f}{\omega^2 - \omega_0^2} 2 \sin \frac{\omega + \omega_0}{2} t \sin \frac{\omega - \omega_0}{2} t. \tag{6}$$

The maximum value of the amplitude is of the order of

$$x_{max} \propto \frac{f}{\omega^2 - \omega_0^2}. \tag{7}$$

However, when $\omega \to \omega_0$, this maximum value diverges. In fact, the limit $\omega \to \omega_0$ is given by the solution

$$x(t) = \frac{f}{2\omega} t \sin \omega t. \tag{8}$$

The amplitude grows linearly and indefinitely.

Of course, such a behavior occurs only in an idealized world with no friction. In the next section we will include friction.

3 Damped Oscillator

Now we include friction proportional to the speed, such as a pendulum moving in honey, or an electric circuit with a capacitor and a coil, together with a resistor. The equation of motion is

$$\ddot{x} + g\dot{x} + \omega_0^2 x = 0. \tag{9}$$

Because the second term represents the friction, we assume $g > 0$.

This equation is easy to solve. Assuming the solution of type $x \propto e^{-i\omega t}$, the equation becomes

$$-\omega^2 - ig\omega + \omega_0^2 = 0. \tag{10}$$

This quadratic equation has solutions

$$\omega = \omega_\pm = \frac{1}{2} \left[-ig \pm \sqrt{4\omega_0^2 - g^2} \right]. \tag{11}$$

In the limit of no friction, the solutions are $\omega = \pm\omega_0$ as expected. However, the friction term gives an “imaginary term” to the frequency, and the oscillation is necessarily damped. Remember that $g > 0$ and $\omega_\pm$ have negative imaginary part. For instance, a simple solution is

$$x(t) = x(0)\frac{1}{2}[e^{-i\omega_+ t} + e^{-i\omega_- t}] = x(0)e^{-gt/2}\cos\sqrt{\omega^2 - \frac{1}{4}g^2}t. \tag{12}$$

Because of the behavior of the solution, the system is called a damped oscillator.

4 Resonances With Friction

Now we exert force on the damped oscillator. The equation of motion is

$$\ddot{x} + g\dot{x} + \omega_0^2 x = f\cos\omega t. \tag{13}$$

There are many ways to solve this equation. Let me use Green's function method.

We first solve Green's equation

$$\ddot{G}(t) + g\dot{G}(t) + \omega_0^2 G(t) = \delta(t-t'). \tag{14}$$

By a Fourier transform,

$$G(t) = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\tilde{G}(\omega)e^{-i\omega t}, \qquad \delta(t-t') = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega(t-t')}, \tag{15}$$

Green's equation becomes

$$(-\omega^2 - ig\omega + \omega_0^2)\tilde{G}(\omega) = e^{i\omega t'}. \tag{16}$$

Therefore,

$$\begin{aligned} G(t) &= \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{e^{i\omega t'}}{-\omega^2 - ig\omega + \omega_0^2}e^{-i\omega t} \\ &= -\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{e^{-i\omega(t-t')}}{(\omega-\omega_+)(\omega-\omega_-)}. \end{aligned} \tag{17}$$

Recall that both $\omega_\pm$ are below the real axis. When $t - t' > 0$, we can close the contour by going back on the inifinite semicircle on the lower half plane, and we pick up the poles at $\omega_\pm$. On the other hand, when $t - t' < 0$, we go back on the upper half plane, and the contour does not encircle either pole. Therefore,

$$G(t) = \theta(t-t')\left[i\frac{e^{-i\omega_+(t-t')} - e^{-i\omega_-(t-t')}}{\omega_+ - \omega_-}\right]. \tag{18}$$

Given this solution, a solution to the Eq. (13) is given as

$$x(t) = \int_0^\infty dt' G(t) f \cos\omega t' = \int_0^t dt' \left[i\frac{e^{-i\omega_+(t-t')} - e^{-i\omega_-(t-t')}}{\omega_+ - \omega_-}\right] f\cos\omega t'. \tag{19}$$

This solution automatically satisfies the boundary conditions $x(0) = \dot{x}(0) = 0$. This is an elementary integral and we find

$$\begin{aligned}
x(t) &= \frac{1}{2} f \frac{i}{\omega_+ - \omega_-} \\
&\quad \left[\frac{e^{-i\omega_+ t} e^{i(\omega_+ + \omega)t'}}{i(\omega_+ + \omega)} + \frac{e^{-i\omega_+ t} e^{i(\omega_+ - \omega)t'}}{i(\omega_+ - \omega)} - \frac{e^{-i\omega_- t} e^{i(\omega_- + \omega)t'}}{i(\omega_- + \omega)} - \frac{e^{-i\omega_+ t} e^{i(\omega_- - \omega)t'}}{i(\omega_- - \omega)}\right]_0^t \\
&= \frac{1}{2} f \frac{i}{\omega_+ - \omega_-} \\
&\quad \left[\frac{e^{i\omega t} - e^{-i\omega_+ t}}{i(\omega_+ + \omega)} + \frac{e^{-i\omega t} - e^{-i\omega_+ t}}{i(\omega_+ - \omega)} - \frac{e^{i\omega t} - e^{-i\omega_- t}}{i(\omega_- + \omega)} - \frac{e^{-i\omega t} - e^{-i\omega_- t}}{i(\omega_- - \omega)}\right] \\
&= -\frac{1}{2} f \left[\frac{e^{i\omega t}}{(\omega_+ + \omega)(\omega_- + \omega)} + \frac{e^{-i\omega t}}{(\omega_+ - \omega)(\omega_- - \omega)}\right] \\
&\quad -f\frac{1}{\omega_+ - \omega_-}\left[\frac{\omega_+ e^{-i\omega_+ t}}{(\omega_+ + \omega)(\omega_+ - \omega)} - \frac{\omega_- e^{-i\omega_- t}}{(\omega_- + \omega)(\omega_- - \omega)}\right].
\end{aligned} \tag{20}$$

In the last expression, the first line represents the inhomogeneous solution and the second line the homogenous one.

It is interesting to see the asymptotic behavior of the solution $t \to \infty$. The second line (the homogenous part) is exponentially damped and can be neglected. What remains is then the first line

$$x_{asym}(t) = -\frac{1}{2} f \left[\frac{e^{i\omega t}}{(\omega_+ + \omega)(\omega_- + \omega)} + \frac{e^{-i\omega t}}{(\omega_+ - \omega)(\omega_- - \omega)}\right]. \tag{21}$$

Recalling Eq. (10),

$$-\omega_\pm^2 - ig\omega_\pm + \omega_0^2 = 0, \tag{22}$$

we find

$$(\omega_+ + \omega)(\omega_- + \omega) = \omega_+\omega_- + (\omega_+ + \omega_-)\omega + \omega^2 = -\omega_0^2 - ig\omega + \omega^2, \qquad (23)$$

and

$$(\omega_+ - \omega)(\omega_- - \omega) = \omega_+\omega_- - (\omega_+ + \omega_-)\omega + \omega^2 = -\omega_0^2 + ig\omega + \omega^2. \qquad (24)$$

Therefore,

$$\begin{aligned} x_{asym}(t) &= -\frac{1}{2}f\left[\frac{e^{i\omega t}}{-\omega_0^2 - ig\omega + \omega^2} + \frac{e^{-i\omega t}}{-\omega_0^2 + ig\omega + \omega^2}\right] \\ &= -\frac{1}{2}f\left[\frac{e^{i\omega t}(-\omega_0^2 + ig\omega + \omega^2) + e^{-i\omega t}(-\omega_0^2 - ig\omega + \omega^2)}{(-\omega_0^2 + \omega^2)^2 + (g\omega)^2}\right] \\ &= -f\left[\frac{(\omega^2 - \omega_0^2)\cos\omega t - g\omega\sin\omega t}{(\omega^2 - \omega_0^2)^2 + (g\omega)^2}\right]. \qquad (25) \end{aligned}$$

Therefore, the amplitude of the oscillator is given by

$$x_{max}(t) = f\frac{1}{\sqrt{(\omega^2 - \omega_0^2)^2 + (g\omega)^2}}. \qquad (26)$$

This is peaked at $\omega = \omega_0$, and quite similar to Breit–Wigner form.

The main difference from quantum mechanics is that you see the resonance in frequency, not in energy. That is because $E = \hbar\omega$ requires $\hbar$, and hence quantum mechanics. But the point that the width in the resonance shape is inversely related to the lifetime in exponential damping is the same.

Therefore, the asymptotic amplitude is a strongly peaked function of ω. Unlike the case without friction, the amplitude does not keep going up even when $\omega = \omega_0$ but rather saturates. You can view it as that the resonant frequency has moved off the real axis to $\omega_\pm$ that cannot be reached by varying ω along the real axis. The strength of the friction determines the width of the peak.

If you keep exerting force for a while and then turn it off, the resonance will be damped exponentially from that point on.