

Electric dipole transitions in Spin network notations.

Basic Definitions

1) Spin network for S_j symbols:

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \begin{array}{c} j_1 \uparrow m_1 \\ \circlearrowleft \\ \swarrow \quad \searrow \\ j_2 \uparrow m_2 \quad j_3 \uparrow m_3 \end{array}$$

$$\begin{array}{c} j_1 \uparrow m_1 \\ \circlearrowleft \\ \swarrow \quad \searrow \\ j_2 \uparrow m_2 \quad j_3 \uparrow m_3 \end{array} = (-1)^{j_1 - m_1} \begin{array}{c} j_1 \downarrow -m_1 \\ \circlearrowright \\ \swarrow \quad \searrow \\ j_2 \uparrow m_2 \quad j_3 \uparrow m_3 \end{array}$$

2) Spin network for spherical harmonics:

$$Y_m^l = Y \begin{array}{c} l \\ \rightarrow \\ m \end{array}$$

$$Y_m^{l*} = Y \begin{array}{c} l \\ \leftarrow \\ m \end{array} = (-1)^{l-m} Y \begin{array}{c} l \\ \rightarrow \\ m \end{array}$$

3) Contractions

$$\sum_m \text{Blob} \begin{array}{c} i \\ \rightarrow \\ m \end{array} \begin{array}{c} j \\ \rightarrow \\ m \end{array} \text{Blob}' = \text{Blob} \text{---} \text{Blob}' \text{ (arrow disappears)}.$$

4) Addition Theorem.

$$\int d\Omega Y_m^l * Y_{m'}^{l'} = \delta_{ll'} \delta_{mm'}$$

$$\Rightarrow \sum_m Y_m^l * Y_m^l = Y^l \frac{1}{Y} Y^l = \frac{(2l+1)}{4\pi}$$

5) Another convention for spherical harmonics.

$$\Theta_m^l = \Theta_{-m}^l = \sqrt{\frac{4\pi}{2l+1}} Y_{-m}^l$$

The addition theorem reads $\Theta^l \Theta = 1$.

e.g. $\Theta_m^1 = \kappa_m/r$. These enjoy $\Theta(\hat{z}) \frac{1}{r} = \delta_{m,0}$

6) Normalization of 3j symbols.

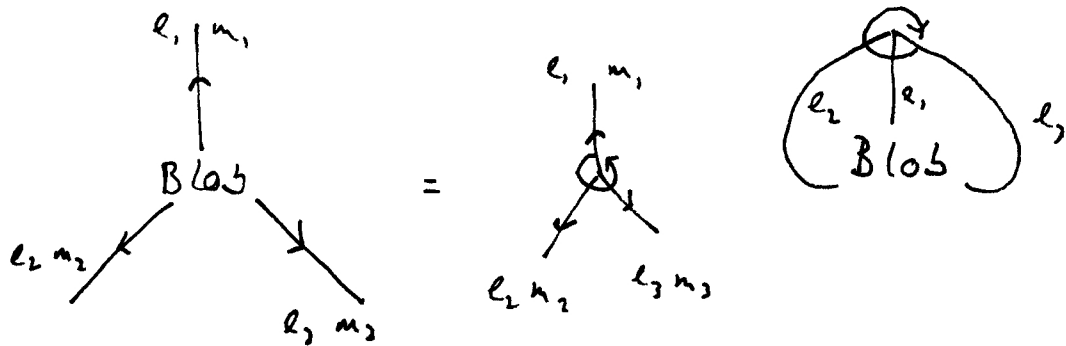
$$1 = \sum_m \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}^2 = \text{graph}$$

(graph = 1).

7) Wigner-Eckart Theorems.

$$\frac{1}{m} \rightarrow \text{Bloch} \frac{1}{m'} = \delta_{mm'} \frac{1}{2l+1} \text{Bloch}$$

7. cont'd.



8. Coupled states

$$\left(\psi_1 \begin{matrix} l_1 m_1 \\ \nearrow \end{matrix}, \psi_2 \begin{matrix} l_2 m_2 \\ \nearrow \end{matrix} \right) \in \mathcal{H}_1 \otimes \mathcal{H}_2$$

$$\psi_1 \begin{matrix} l_1 \\ \nearrow \end{matrix} \otimes \psi_2 \begin{matrix} l_2 \\ \nearrow \end{matrix} \rightarrow \begin{matrix} L \\ \rightarrow \end{matrix} \sqrt{2L+1} \text{ is a normalized state in } \mathcal{H}_1 \otimes \mathcal{H}_2$$

Check using 1st form of W-E theorem

9. Integral of three-spherical harmonics

$$I = \int d\Omega \Theta \begin{matrix} l_1 m_1 \\ \nearrow \end{matrix} \Theta \begin{matrix} l_2 m_2 \\ \nearrow \end{matrix} \Theta \begin{matrix} l_3 m_3 \\ \nearrow \end{matrix}$$

$$= \begin{matrix} l_1 m_1 \\ \nearrow \end{matrix} \int d\Omega \begin{matrix} l_1 \\ \nearrow \end{matrix} \begin{matrix} l_2 \\ \nearrow \end{matrix} \begin{matrix} l_3 \\ \nearrow \end{matrix}$$

But $\Theta \begin{matrix} l_1 \\ \nearrow \end{matrix} \Theta \begin{matrix} l_2 \\ \nearrow \end{matrix} \Theta \begin{matrix} l_3 \\ \nearrow \end{matrix}$ is a rotational invariant so choose $\hat{n} = \hat{z}$, $\forall \ell \quad E_{z\ell}(\hat{z}) = \int d\Omega$

$$I = \sqrt{\pi} \begin{matrix} l_1 m_1 \\ \nearrow \end{matrix} \cdot \begin{matrix} l_1 0 \\ \nearrow \end{matrix}$$

Example: Dipole transition

Transition rate is

$g = \text{photon momentum.}$

$|i\rangle \rightarrow |f\rangle.$

$$\int \frac{d\Omega_g}{(2\pi\hbar)^3} \left| \frac{1}{g} \right|^2 \frac{(2\nu)^2 |g|^4}{\hbar} |\langle f | \vec{\epsilon}_\lambda \cdot \vec{D} | i \rangle|^2$$

In hydrogen

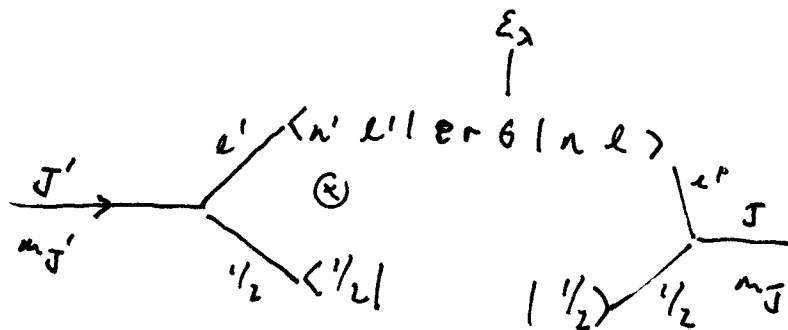
$$|i\rangle = \begin{array}{l} |n, l\rangle \\ \otimes \\ |1/2\rangle \end{array} \begin{array}{l} e \\ \otimes \\ 1/2 \end{array} \xrightarrow[m_J]{J} \sqrt{2J+1}$$

$$\langle f | = \begin{array}{l} \langle n', l' | \\ \otimes \\ \langle 1/2 | \end{array} \xrightarrow[m_{J'}]{J'} \sqrt{2J'+1}$$

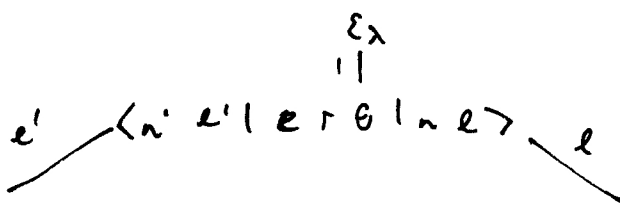
$$\vec{\epsilon}_\lambda \cdot \vec{D} = e \cdot \vec{r} \epsilon_\lambda \frac{1}{r}$$

The amplitude is

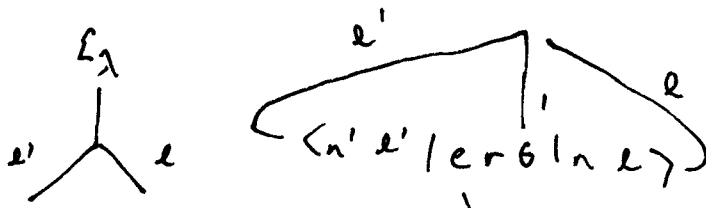
$$\langle f | \vec{\epsilon}_\lambda \cdot \vec{D} | i \rangle = \sqrt{2J+1} \sqrt{2J'+1}$$



The

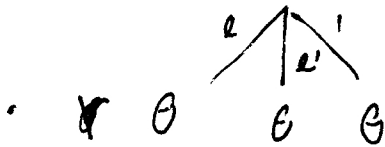


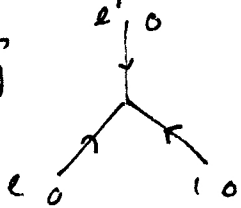
gives



The reduced matrix element is

$$\int e \int r^2 dr r \psi_{n'l'}(r) \psi_{nl}(r) d\Omega \frac{\sqrt{(2l'+1)(2l+1)}}{4\pi}$$



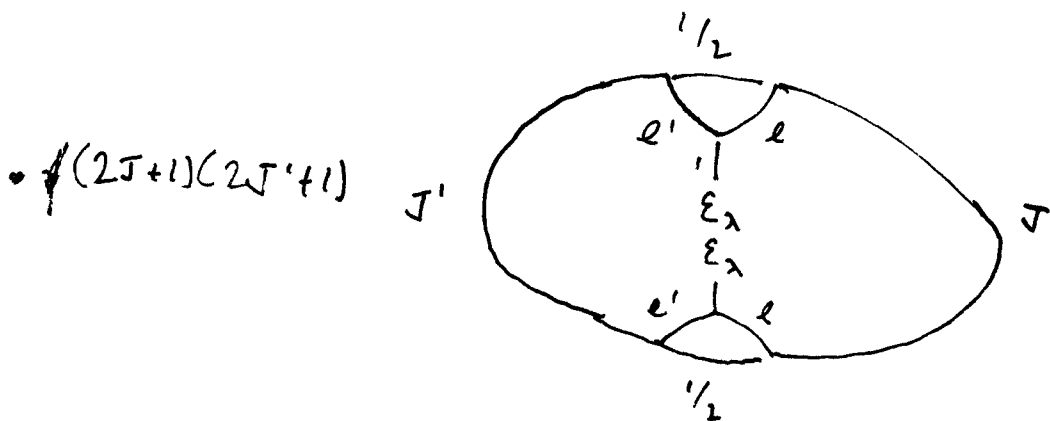
$$= e \sqrt{(2l'+1)(2l+1)} \int r^2 dr r \psi_{nl}(r) \psi_{n'l'}(r)$$


$$:= \underline{R}$$

Now, the ^{rates} squared amplitude is

$$W_i = \sum_f \frac{1}{(2J+1)} \sum_i \sum_\lambda \int \frac{dR_g}{(2\pi t)^3} g^2 \frac{(2\pi)^2 |\vec{q}'|}{t} |\langle f | \vec{\epsilon}_\lambda \cdot \vec{D} | i \rangle|^2$$

$$= \frac{1}{(2J+1)} \sum_\lambda \int \frac{dR_g}{(2\pi t)^3} g^2 \frac{(2\pi)^2 |\vec{q}'|}{t}$$



Do polarization sum:

$\epsilon_\lambda \epsilon_\lambda =$ projector onto space orthogonal to \vec{q}

$=$ 3×3 matrix $= (I - \vec{p} \vec{p})$

$$= \frac{2}{3} I + \underbrace{\left(\frac{1}{3} - \vec{p} \vec{p} \right)}$$

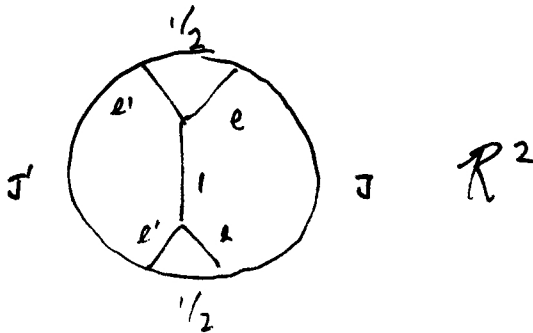
Traceless part: Transforms as $\ell=2$.

and $\int dR_g$ of it vanishes.

Thus

$$W_i = (2J'+1) \cdot \frac{4\pi}{(2\pi\hbar)^3} |g|^3 \cdot \frac{(2\pi)^2}{\hbar} \cdot \frac{2}{3}$$

$\int d\Omega_g$ (above the fraction) \int Polarization (above the last fraction)



~~scribble~~

But

⇒ Final answer:

$$W_i = (2J'+1) \cdot \frac{2}{\hbar^4} |g|^3 \cdot \frac{2}{3} \left[\text{Circular Diagram with } R \right]^2$$

where, again,

$$R = e^{\sqrt{(2l+1)(2l'+1)}} \int r^2 dr \cdot \psi_{nl}(r) \psi_{n'l'}(r)$$