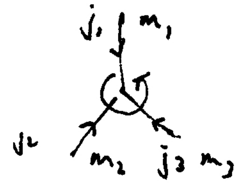
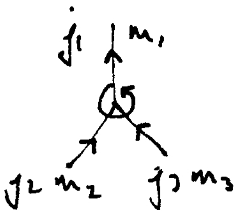
Electric dipole transitions in Spin network notations.

Basic Definitions

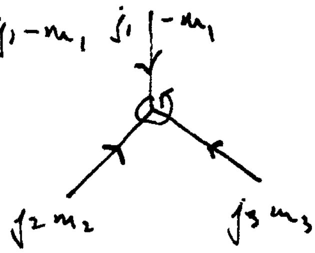
1) Spin network for 3j symbol:

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} =$$





$$= (-1)^{j_1 - m_1}$$



2) Spin network for spherical harmonics:

$$Y^{\ell}_{m} = Y \xrightarrow[m]{\ell}$$

$$Y^{\ell\,*}_{m} = Y \xleftarrow[m]{\ell} = (-1)^{\ell - m}\, Y \xrightarrow[m]{\ell}.$$

3) Contractions

$$\sum_m \text{Blob} \xrightarrow[m]{j} \xrightarrow[m]{j} \text{Blob}' = \text{Blob} \longrightarrow \text{Blob}' \text{ (arrow disappears).}$$

4) Addition Theorem.

$$\int d\Omega \, Y^{\ell *}_{m} Y^{\ell'}_{m'} = \delta_{\ell\ell'}\delta_{mm'}$$

$$\Longrightarrow \sum_m Y^{\ell *}_{m} Y^{\ell}_{m} = Y \overset{\ell}{\text{——}} Y = \frac{(2\ell+1)}{4\pi}.$$

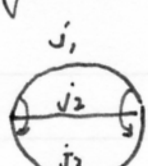
5) Another convention for spherical harmonics.

$$\Theta^{\ell}_{m} = \Theta \overset{\ell}{\nearrow}_{m} = \sqrt{\frac{4\pi}{2\ell+1}}\, Y \overset{\ell}{\nearrow}_{m}$$

The addition theorem reads $\Theta \overset{\ell}{\text{——}} \Theta = 1.$

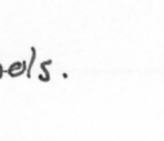
e.g. $\Theta^{1}_{m} = x_m / r$. These enjoy $\Theta(\hat{z}) \xrightarrow[m]{\ell} = \delta_{m,0}$

6) Normalization of 3j symbols.

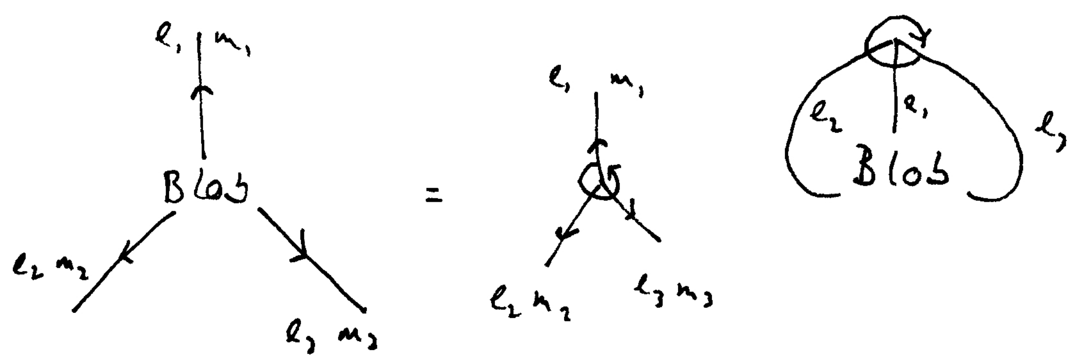
$$1 = \sum_m \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}^2 =$$ 

(Θ graph = 1).

7) Wigner-Eckart theorems.

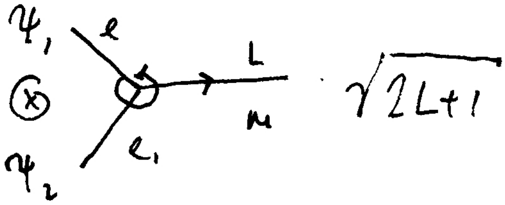
$$\xrightarrow[m]{\ell} \text{Blob} \xrightarrow[m']{\ell} = \delta_{mm'} \frac{1}{2\ell+1}$$ 

7. cont'd.



8. Coupled states

$$\left(\psi_1^{\ell m},\ \psi_2^{\ell' m'} \right) \in \mathcal{H}_1 \otimes \mathcal{H}_2.$$

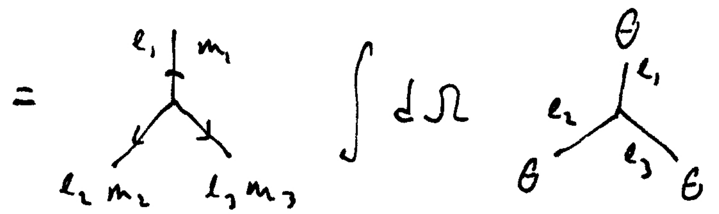


$\sqrt{2L+1}$ is a normalized state in $\mathcal{H}_1 \otimes \mathcal{H}_2$

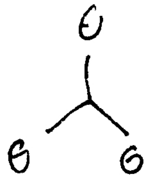
Check using 1st form of W-E theorem

9 Integral of three-spherical Harmonics

$$I = \int d\Omega\ \Theta^{\ell_1 m_1}\ \Theta^{\ell_2 m_2}\ \Theta^{\ell_3 m_3}$$

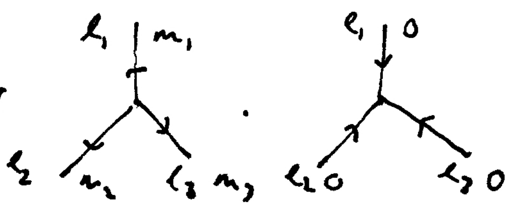


But



is a rotational invariant so choose $\hat{n} = \hat{z}$, use $\Theta_{\ell m}(\hat{z}) = \delta_{m0}$

$I = 4\pi$



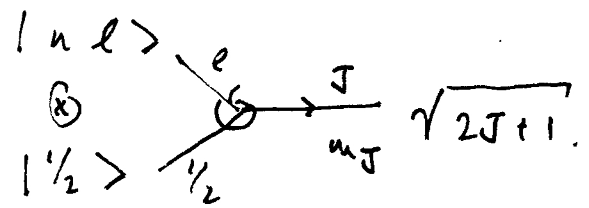
Example: Dipole transition

Transition rate is $\quad\quad q$ = photon momentum.

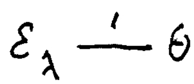
$|i\rangle \to |f\rangle$.

$$\int \frac{d\Omega_q}{(2\pi\hbar)^3}\, |\vec{q}^{\,2}|\, \frac{(2\pi)^2\, |\vec{q}|}{\hbar}\, |\langle f|\, \vec{\varepsilon}_\lambda \cdot \vec{D}\, |i\rangle|^2.$$

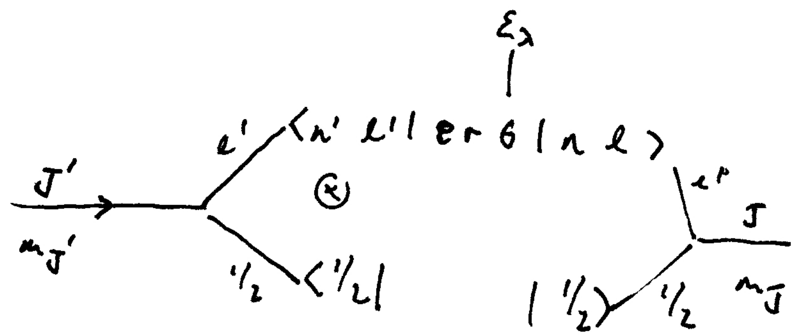
In hydrogen

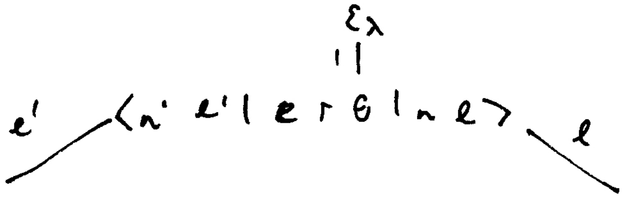
$|i\rangle =$ 

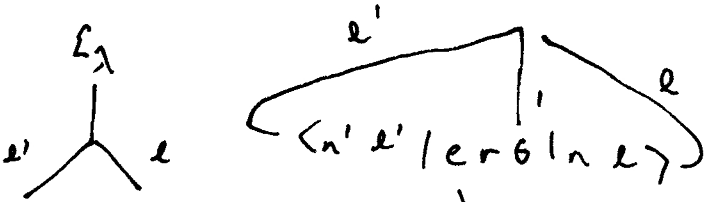
$\langle f| =$ 

$\vec{\varepsilon}_\lambda \cdot \vec{D} = e.\, r$ 

The amplitude is

$\langle f|\, \vec{\varepsilon}_\lambda \cdot \vec{D}\, |i\rangle = \sqrt{2J+1}\sqrt{2J'+1}$ 

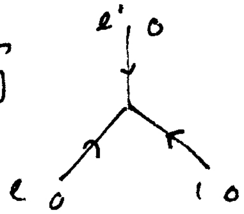
The 

gives 

The reduced matrix element is

$$e \int r^2 dr \; r \, \psi_{n'l'}(r) \, \psi_{nl}(r) \, d\Omega \sqrt{\frac{(2l'+1)(2l+1)}{4\pi}}$$

$\cdot \; Y \; \theta$  $\theta \quad \theta$

$= e \sqrt{(2l'+1)(2l+1)}$  $\int r^2 dr \; r \, \psi_{nl}(r) \, \psi_{n'l'}(r)$

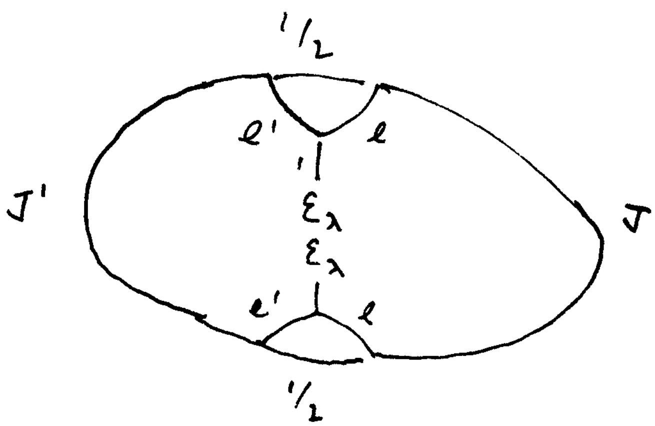
$:= \underline{R}.$

Now, the ~~squared amplitude~~ rates is

$$W_i = \sum_f \frac{1}{(2J+1)} \sum_i \sum_\lambda \int \frac{d\Omega_q}{(2\pi\hbar)^3}\, \vec{q}^{\,2} \frac{(2\pi)^2 |\vec{q}|}{\hbar} \left| \langle f | \vec{\varepsilon}_\lambda \cdot \vec{D} | i \rangle \right|^2$$

$$= \frac{1}{(2J+1)} \sum_\lambda \int \frac{d\Omega_q}{(2\pi\hbar)^3}\, q^2 \frac{(2\pi)^2 |\vec{q}|}{\hbar}$$

$$\cdot \cancel{f}(2J+1)(2J'+1)$$



Do polarization sum:

$$—\varepsilon_\lambda \varepsilon_\lambda— = \text{Projector onto space orthogonal to } \vec{q}$$

$$= 3\times 3 \text{ matrix} = (I - \vec{p}\,\vec{p})$$

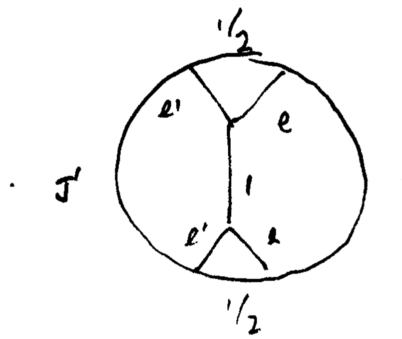
$$= \frac{2}{3} I + \underbrace{\left(\frac{1}{3} - \vec{p}\,\vec{p}\right)}$$

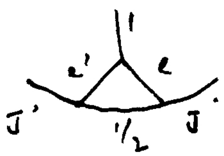
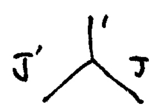
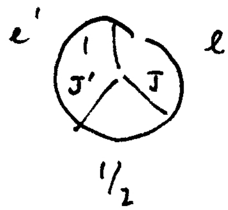
Traceless part: Transforms as $l = 2$.

and $\int d\Omega_q$ of it vanishes.

Thus

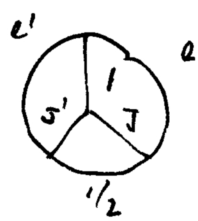
$$W_i = (2J'+1) \cdot \underbrace{\frac{4\pi}{(2\pi\hbar)^3}}_{d\Omega_q} |q|^3 \cdot \frac{(2\pi)^2}{\hbar} \cdot \underbrace{\frac{2}{3}}_{\text{Polarization.}}$$

$\cdot$  R^2

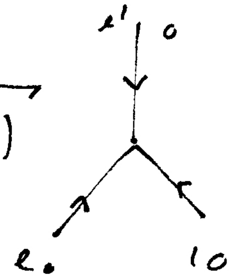
But  =  

and  $= 1.$

$\Rightarrow$ Final answer:

$$W_i = (2J'+1) \cdot \frac{2}{\hbar^4} q^3 \cdot \frac{2}{3}\Big[$$  $$R\Big]^2$$

Where, again,

$R = e\sqrt{(2l+1)(2l'+1)}$  $\int r^2 dr \, r \, \psi_{nl}(r) \psi_{n'l'}(r)$