HW #12

1.

(a)

Using the given mode expansion of the vector potential, we check the Coulomb gauge condition. Acting the divergence on the vector potential simply pulls out $i\vec{p}/\hbar$ for the term with annihilation operators or $-i\vec{p}/\hbar$ for the term with creation operators. Therefore, all we need to show is that $\vec{p}\cdot\vec{\epsilon}_\pm(\vec{p}) = 0$. Here, I emphasized that the polarization vector depends on the momentum (its direction).

The momentum vector is $\vec{p} = p(\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta)$, while the two (linear) polarization vectors are given by
$\vec{\epsilon}_1(\vec{p}) = (\cos\theta\cos\phi,\ \cos\theta\sin\phi,\ -\sin\theta)$
$\vec{\epsilon}_2(\vec{p}) = (-\sin\phi,\ \cos\phi,\ 0)$
It is straightforward to check
$\vec{p}\cdot\vec{\epsilon}_1(\vec{p}) = p(\sin\theta\cos\phi\cos\theta\cos\phi + \sin\theta\sin\phi\cos\theta\sin\phi - \cos\theta\sin\theta) = 0,$
$\vec{p}\cdot\vec{\epsilon}_2(\vec{p}) = p(-\sin\theta\cos\phi\ \sin\phi + \sin\theta\sin\phi\cos\phi) = 0.$

The polarization vectors for cirular polarization (helicity eigenstates) are linear combinations of $\vec{\epsilon}_{1,2}$ and hence they are orthogonal to the momentum vector as well.

Therefore, the given mode expansion satisfies the Coulomb gauge condition.

(b)

In the Coulomb gauge, the scalar potential vanishes in the absence of electric charges, and the electric field is simply $\vec{E} = \frac{1}{c}\dot{\vec{A}}$. On the other hand, we can use integration by parts and the Coulomb gauge condition to simplify the term
$\int d\vec{x}\,\vec{B}^2 = \int d\vec{x}(\vec{\nabla}\times\vec{A})\cdot(\vec{\nabla}\times\vec{A}) = -\int d\vec{x}\,\vec{A}\cdot\Delta\vec{A}$
Therefore,

$$H = \frac{1}{8\pi}\int d\vec{x}\left(\frac{1}{c^2}\dot{\vec{A}}^2 - \vec{A}\cdot\Delta\vec{A}\right)$$

For the first term,

$$\frac{1}{8\pi}\int d\vec{x}\,\frac{1}{c^2}\dot{\vec{A}}^2$$

$$= \frac{1}{8\pi}\int d\vec{x}\,\frac{1}{c^2}\left(\sqrt{\frac{2\pi\hbar c^2}{L^3}}\sum_{\vec{p}}\sqrt{\omega_p}\sum_{\pm}\left(-i\,\vec{\epsilon}_\pm(\vec{p})\,a_\pm(\vec{p})\,e^{i\vec{p}\cdot\vec{x}/\hbar} + i\,\vec{\epsilon}_\pm^{\,*}(\vec{p})\,a_\pm^\dagger(\vec{p})\,e^{i\vec{p}\cdot\vec{x}/\hbar}\right)\right)^2$$

$$= \frac{\hbar}{4L^3}\sum_{\vec{p},\vec{q}}\sqrt{\omega_p\,\omega_q}\sum_{\lambda,\lambda'}\int d\vec{x}$$
$$\left(-i\,\vec{\epsilon}_\lambda(\vec{p})\,a_\lambda(\vec{p})\,e^{i\vec{p}\cdot\vec{x}/\hbar} + i\,\vec{\epsilon}_\lambda(\vec{p})^*\,a_\lambda(\vec{p})^\dagger\,e^{i\vec{p}\cdot\vec{x}/\hbar}\right)\left(-i\,\vec{\epsilon}_{\lambda'}(\vec{q})\,a_{\lambda'}(\vec{q})\,e^{i\vec{q}\cdot\vec{x}/\hbar} + i\,\vec{\epsilon}_{\lambda'}(\vec{q})^*\,a_{\lambda'}(\vec{q})^\dagger\,e^{i\vec{q}\cdot\vec{x}/\hbar}\right)$$
$$= \frac{\hbar}{4L^3}\sum_{\vec{p},\vec{q}}\sqrt{\omega_p\,\omega_q}\sum_{\lambda,\lambda'}L^3$$
$$\left(\vec{\epsilon}_\lambda(\vec{p})\cdot\vec{\epsilon}_{\lambda'}(\vec{q})^*\,a_\lambda(\vec{p})\,a_{\lambda'}^\dagger(\vec{q})\,\delta_{\vec{p},\vec{q}} + \vec{\epsilon}_\lambda(\vec{p})^*\cdot\vec{\epsilon}_{\lambda'}(\vec{q})\,a_\lambda^\dagger(\vec{p})\,a_{\lambda'}(\vec{q})\right.$$
$$\left.-\vec{\epsilon}_\lambda(\vec{p})\cdot\vec{\epsilon}_{\lambda'}(\vec{q})\,a_\lambda(\vec{p})\,a_{\lambda'}(\vec{q})\,\delta_{\vec{p},-\vec{q}} - \vec{\epsilon}_\lambda(\vec{p})^*\cdot\vec{\epsilon}_{\lambda'}(\vec{q})^*\,a_\lambda^\dagger(\vec{p})\,a_{\lambda'}^\dagger(\vec{q})\,\delta_{\vec{p},-\vec{q}}\right)$$
$$= \frac{\hbar}{4}\sum_{\vec{p}}\omega_p\sum_{\lambda,\lambda'}$$
$$\left(\vec{\epsilon}_\lambda(\vec{p})\cdot\vec{\epsilon}_{\lambda'}(\vec{p})^*\,a_\lambda(\vec{p})\,a_{\lambda'}^\dagger(\vec{p}) + \vec{\epsilon}_\lambda(\vec{p})^*\cdot\vec{\epsilon}_{\lambda'}(\vec{p})\,a_\lambda^\dagger(\vec{p})\,a_{\lambda'}(\vec{p})\right.$$
$$\left.-\vec{\epsilon}_\lambda(\vec{p})\cdot\vec{\epsilon}_{\lambda'}(-\vec{p})\,a_\lambda(\vec{p})\,a_{\lambda'}(-\vec{p}) - \vec{\epsilon}_\lambda(\vec{p})^*\cdot\vec{\epsilon}_{\lambda'}(-\vec{p})^*\,a_\lambda^\dagger(\vec{p})\,a_{\lambda'}^\dagger(-\vec{p})\right)$$

Similarly for the second term,

$$-\frac{1}{8\pi}\int d\vec{x}\,\vec{A}\cdot\Delta\vec{A}$$

$$= -\frac{1}{8\pi}\,\frac{2\pi\hbar c^2}{L^3}\int d\vec{x}\left(\sum_{\vec{p}}\frac{1}{\sqrt{\omega_p}}\sum_\lambda\left(\vec{\epsilon}_\lambda(\vec{p})\,a_\lambda(\vec{p})\,e^{i\vec{p}\cdot\vec{x}/\hbar} + \vec{\epsilon}_\lambda^{\,*}(\vec{p})\,a_\lambda^\dagger(\vec{p})\,e^{i\vec{p}\cdot\vec{x}/\hbar}\right)\right)\cdot$$
$$\Delta\left(\sum_{\vec{q}}\frac{1}{\sqrt{\omega_q}}\sum_{\lambda'}\left(\vec{\epsilon}_{\lambda'}(\vec{q})\,a_{\lambda'}(\vec{q})\,e^{i\vec{q}\cdot\vec{x}/\hbar} + \vec{\epsilon}_{\lambda'}^{\,*}(\vec{q})\,a_{\lambda'}^\dagger(\vec{q})\,e^{i\vec{q}\cdot\vec{x}/\hbar}\right)\right)$$
$$= \frac{\hbar c^2}{4L^3}\int d\vec{x}\left(\sum_{\vec{p}}\frac{1}{\sqrt{\omega_p}}\sum_\lambda\left(\vec{\epsilon}_\lambda(\vec{p})\,a_\lambda(\vec{p})\,e^{i\vec{p}\cdot\vec{x}/\hbar} + \vec{\epsilon}_\lambda^{\,*}(\vec{p})\,a_\lambda^\dagger(\vec{p})\,e^{i\vec{p}\cdot\vec{x}/\hbar}\right)\right)\cdot$$
$$\left(\sum_{\vec{q}}\frac{1}{\sqrt{\omega_q}}\,\frac{q^2}{\hbar^2}\sum_{\lambda'}\left(\vec{\epsilon}_{\lambda'}(\vec{q})\,a_{\lambda'}(\vec{q})\,e^{i\vec{q}\cdot\vec{x}/\hbar} + \vec{\epsilon}_{\lambda'}^{\,*}(\vec{q})\,a_{\lambda'}^\dagger(\vec{q})\,e^{i\vec{q}\cdot\vec{x}/\hbar}\right)\right)$$
$$= \frac{c^2}{4\hbar}\sum_{\vec{p}}\frac{q^2}{\omega_p}\sum_{\lambda,\lambda'}\left(\vec{\epsilon}_\lambda(\vec{p})\cdot\vec{\epsilon}_{\lambda'}^{\,*}(\vec{p})\,a_\lambda(\vec{p})\,a_{\lambda'}^\dagger(\vec{p}) + \vec{\epsilon}_\lambda^{\,*}(\vec{p})\cdot\vec{\epsilon}_{\lambda'}(\vec{p})\,a_\lambda^\dagger(\vec{p})\,a_{\lambda'}(\vec{p})\right.$$
$$\left.+\vec{\epsilon}_\lambda(\vec{p})\cdot\vec{\epsilon}_{\lambda'}(-\vec{p})\,a_\lambda(\vec{p})\,a_{\lambda'}(-\vec{p}) + \vec{\epsilon}_\lambda^{\,*}(\vec{p})\cdot\vec{\epsilon}_{\lambda'}^{\,*}(-\vec{p})\,a_\lambda^\dagger(\vec{p})\,a_{\lambda'}^\dagger(-\vec{p})\right)$$

Because $\omega_q = c\,q/\hbar$, $\frac{c^2}{4\hbar}\,\frac{q^2}{\omega_q} = \frac{\hbar\,\omega_q}{4}$, the sum of two terms cancel pieces with two annihilation operators or two creation operators. The total Hamiltonian is

$$H = \sum_{\vec{p}} \frac{\hbar\omega_p}{2} \sum_{\lambda,\lambda'} \left(\vec{\epsilon}_\lambda(\vec{p})\cdot\vec{\epsilon}_{\lambda'}(\vec{p})^* a_\lambda(\vec{p}) a_{\lambda'}{}^\dagger(\vec{p}) + \vec{\epsilon}_\lambda(\vec{p})^* \cdot \vec{\epsilon}_{\lambda'}(\vec{p}) a_\lambda{}^\dagger(\vec{p}) a_{\lambda'}(\vec{p})\right)$$

Using the orthonormality of the polarization vectors, $\vec{\epsilon}_\lambda(\vec{p})^* \cdot \vec{\epsilon}_{\lambda'}(\vec{p}) = \delta_{\lambda,\lambda'}$, it further simplifies to

$$H = \sum_{\vec{p}} \frac{\hbar\omega_p}{2} \sum_\lambda \left(a_\lambda(\vec{p}) a_\lambda{}^\dagger(\vec{p}) + a_\lambda{}^\dagger(\vec{p}) a_\lambda(\vec{p})\right)$$

$$= \sum_{\vec{p},\lambda} \hbar\omega_p\left(a_\lambda{}^\dagger(\vec{p}) a_\lambda(\vec{p}) + \frac{1}{2}\right)$$

We find that the Hamiltonian is nothing but an infinite collection of harmonic oscillators of definite momentum and helicity.

(c)

The only piece of Hamitonian we need is $H = \hbar\omega_p a_+{}^\dagger(\vec{p}) a_+(\vec{p})$ for the mode we consider, because other terms vanish when acted on the vacuum $|0\rangle$ once we decide to neglect the zero-point piece. We suppress the subscript + and the momentum argument to simplify the expressions. The l.h.s. of the equation is

$$i\hbar \frac{d}{dt} |f e^{-icpt/\hbar}\rangle = i\hbar \frac{d}{dt} e^{-|f|^2/2} e^{f e^{-icpt/\hbar} a^\dagger} |0\rangle$$

$$= e^{-|f|^2/2} f c p e^{-icpt/\hbar} a^\dagger e^{f e^{-icpt/\hbar} a^\dagger} |0\rangle$$

$$= f \hbar\omega a^\dagger e^{-icpt/\hbar} |f e^{-icpt/\hbar}\rangle$$

The r.h.s. is

$$H |f e^{-icpt/\hbar}\rangle = \hbar\omega a^\dagger a |f e^{-icpt/\hbar}\rangle$$

$$= \hbar\omega a^\dagger f e^{-icpt/\hbar} |f e^{-icpt/\hbar}\rangle$$

Both sides agree.

(d)

Given the state $|f, t\rangle = |f e^{-icpt/\hbar}\rangle$, we calculate the expectation value of the vector potential,

$$\left\langle f, t \middle| \vec{A}(\vec{x}, t) \middle| f, t\right\rangle = \left\langle f e^{-icpt/\hbar} \middle| \sqrt{\frac{2\pi\hbar c^2}{L^3}} \sum_{\vec{p}} \frac{1}{\sqrt{\omega_p}} \sum_\pm \left(\vec{\epsilon}_\pm(\vec{p}) a_\pm(\vec{p}) e^{i\vec{p}\cdot\vec{x}/\hbar} + \vec{\epsilon}_\pm{}^*(\vec{p}) a_\pm{}^\dagger(\vec{p}) e^{i\vec{p}\cdot\vec{x}/\hbar} \middle| f e^{-icpt/\hbar}\right\rangle\right.$$

$$= \sqrt{\frac{2\pi\hbar c^2}{L^3}} \frac{1}{\sqrt{\omega_p}} \left(\vec{\epsilon}_+(\vec{p}) f e^{-icpt/\hbar} e^{i\vec{p}\cdot\vec{x}/\hbar} + \vec{\epsilon}_+{}^*(\vec{p}) f^* e^{icpt/\hbar} e^{-i\vec{p}\cdot\vec{x}/\hbar}\right)$$

This is nothing but the plane wave solution to the Maxwell equation in the Coulomb gauge $\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta\right)\vec{A} = 0$. It is a coherent electromagnetic wave, and indeed the laser is described by this state.

For example, let us take $\vec{p} = (0, 0, p)$, namely $\theta = 0$, $\phi = 0$. Then, $\vec{\epsilon}_+ = \frac{1}{\sqrt{2}}(1, i, 0)$. We also take f real. Then the expectation value is

$$\left\langle f, t \middle| A_x(\vec{x}, t) \middle| f, t\right\rangle = \sqrt{\frac{2\pi\hbar c^2}{L^3}} \frac{1}{\sqrt{\omega_p}} \sqrt{2} f \cos(p(ct - z)/\hbar)$$

$$\left\langle f, t \middle| A_y(\vec{x}, t) \middle| f, t\right\rangle = \sqrt{\frac{2\pi\hbar c^2}{L^3}} \frac{1}{\sqrt{\omega_p}} \sqrt{2} f \sin(p(ct - z)/\hbar)$$

$$\left\langle f, t \middle| A_z(\vec{x}, t) \middle| f, t\right\rangle = 0$$

The electric field is simply its time derivative (devided by c). It indeed is circularly polarized light of frequency $c p/\hbar$ and wave vector $p/\hbar$.

In other words, laser is the Bose–Einstein condensate of photons.