Take-Home Final Exam

1. $3\,d \to 2\,p$ decays

(a)

In the electric dipole (E1) transitions, a photon is emitted in the $J = 1$ state, and hence only possible values of J_z of the photon are $J_z = +\hbar,\ 0,\ -\hbar$. The initial state has $J_z = m\,\hbar = 2\,\hbar$. Suppose the final state of the atom has $J_z = m'\,\hbar$. Then the possible values of the total J_z are: $J_z = (m' + 1)\,\hbar,\ m'\,\hbar,\ (m' - 1)\,\hbar$. On the other hand, because the final state of the atom is the $2\,p$ state, the only possible values of m' are $m' = +1, 0, -1$. The only way to conserve J_z is to have $m' = 1$ and the photon in $J_z = +\hbar$ state.

Another way of arriving at the same conclusion is to consider the matrix element of the electric dipole operator between the atomic states $\left\langle 2\,p \,\middle|\, e\,\vec{x} \,\middle|\, 3\,d,\ m = +2 \right\rangle$. Because $e\,\vec{x}$ is a tensor operator with $q = 1$, the matrix element is proportional to the Clebsch-Gordan coefficient $\langle l = 1,\ m_l \,|\, l = 2,\ q = 1;\ m = +2,\ k\rangle$. In order for it not to vanish, we need $m_l = 2 + k$, where $m_l = 1, 0, -1$ and $k = 1, 0, -1$. The only possibility is $m_l = 1$, $k = -1$.

(Comparing the two arguments, it is clear that the tensor operator with $k = -1$ corresponds to the emission of a photon in the $J_z = +\hbar$ state.)

(b)

We use Eq. (59) of the lecture notes. From the part (a), we know that we only need to consider $m = +1$ final state.

$$W_i = \int \frac{d\vec{q}}{(2\pi\hbar)^3}\ \frac{2\pi c q}{\hbar} \sum_\lambda \left| \vec{\epsilon}_\lambda^{\,*}(\vec{q}) \cdot \left\langle 2\,p,\, m = +1 \,\middle|\, \vec{D} \,\middle|\, 3\,d,\ m = +2 \right\rangle \right|^2 2\pi\,\delta(E_f - E_i)$$

Let us first calculate the matrix element of $\vec{D} = e\,\vec{x}$. In the position representation, the wave functions of the hydrogen levels are

$$\mathbf{R = \frac{2}{n^2} \left(\frac{(n-l-1)!}{a^3\,(n+l)!}\right)^{1/2} E^{-r/(n\,a)} \left(\frac{2\,r}{n\,a}\right)^l \text{LaguerreL}\left[n-l-1,\ 2\,l+1,\ \frac{2\,r}{n\,a}\right]}$$

$$\frac{2^{1+l}\, e^{-\frac{r}{a\,n}} \left(\frac{r}{a\,n}\right)^l \sqrt{\frac{(-1-l+n)!}{a^3\,(l+n)!}}\ \text{LaguerreL}\left[-1-l+n,\ 1+2\,l,\ \frac{2\,r}{a\,n}\right]}{n^2}$$

$$\boldsymbol{\psi = \mathbf{R}\ \text{SphericalHarmonicY}[l,\ m,\ \theta,\ \phi]}$$

$$\frac{1}{n^2}\left(2^{1+l}\, e^{-\frac{r}{a\,n}} \left(\frac{r}{a\,n}\right)^l \sqrt{\frac{(-1-l+n)!}{a^3\,(l+n)!}}\ \text{LaguerreL}\left[-1-l+n,\ 1+2\,l,\ \frac{2\,r}{a\,n}\right] \text{SphericalHarmonicY}[l,\ m,\ \theta,\ \phi]\right)$$

$$\psi \,/.\, \{n \to 3,\ l \to 2,\ m \to 2\}$$

$$\frac{\sqrt{\frac{1}{a^3}}\ e^{-\frac{r}{3a}+2 i \phi}\ r^2\ \mathrm{Sin}[\theta]^2}{162\ a^2\ \sqrt{\pi}}$$

$$\mathrm{Integrate}\left[\mathrm{Integrate}\left[\left(\frac{\sqrt{\frac{1}{a^3}}\ e^{-\frac{r}{3a}}\ r^2\ \mathrm{Sin}[\theta]^2}{162\ a^2\ \sqrt{\pi}}\right)^2 2\pi\ \mathrm{Sin}[\theta],\ \{\theta, 0, \pi\}\right] r^2,\ \{r, 0, \infty\},\ \mathrm{Assumptions} \to \mathrm{Re}[a] > 0\right]$$

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$$\psi \,/.\, \{n \to 2,\ l \to 1,\ m \to 1\}$$

$$-\frac{\sqrt{\frac{1}{a^3}}\ e^{-\frac{r}{2a}+i \phi}\ r\ \mathrm{Sin}[\theta]}{8\ a\ \sqrt{\pi}}$$

$$\mathrm{Integrate}\left[\mathrm{Integrate}\left[\left(-\frac{\sqrt{\frac{1}{a^3}}\ e^{-\frac{r}{2a}}\ r\ \mathrm{Sin}[\theta]}{8\ a\ \sqrt{\pi}}\right)^2 2\pi\ \mathrm{Sin}[\theta],\ \{\theta, 0, \pi\}\right] r^2,\ \{r, 0, \infty\},\ \mathrm{Assumptions} \to \mathrm{Re}[a] > 0\right]$$

1

$\langle 2p,\ m = 1 \,|\, x \,|\, 3d,\ m = 2\rangle$

$$\mathrm{Integrate}\left[\mathrm{Integrate}\left[\mathrm{Integrate}\left[-\frac{\sqrt{\frac{1}{a^3}}\ e^{-\frac{r}{2a}-i \phi}\ r\ \mathrm{Sin}[\theta]}{8\ a\ \sqrt{\pi}}\ \frac{\sqrt{\frac{1}{a^3}}\ e^{-\frac{r}{3a}+2 i \phi}\ r^2\ \mathrm{Sin}[\theta]^2}{162\ a^2\ \sqrt{\pi}}\ r\ \mathrm{Sin}[\theta]\ \mathrm{Cos}[\phi],\ \{\phi, 0, 2\pi\}\right] r^2\ \mathrm{Sin}[\theta],\ \{\theta, 0, \pi\}\right],\ \{r, 0, \infty\},\ \mathrm{Assumptions} \to \mathrm{Re}[a] > 0\right]$$

$$-\frac{165888\ a}{78125}$$

$\langle 2p,\ m = 1 \,|\, y \,|\, 3d,\ m = 2\rangle$

$$\mathrm{Integrate}\left[\mathrm{Integrate}\left[\mathrm{Integrate}\left[-\frac{\sqrt{\frac{1}{a^3}}\ e^{-\frac{r}{2a}-i \phi}\ r\ \mathrm{Sin}[\theta]}{8\ a\ \sqrt{\pi}}\ \frac{\sqrt{\frac{1}{a^3}}\ e^{-\frac{r}{3a}+2 i \phi}\ r^2\ \mathrm{Sin}[\theta]^2}{162\ a^2\ \sqrt{\pi}}\ r\ \mathrm{Sin}[\theta]\ \mathrm{Sin}[\phi],\ \{\phi, 0, 2\pi\}\right] r^2\ \mathrm{Sin}[\theta],\ \{\theta, 0, \pi\}\right],\ \{r, 0, \infty\},\ \mathrm{Assumptions} \to \mathrm{Re}[a] > 0\right]$$

$$-\frac{165888\ i\ a}{78125}$$

$\langle 2p,\ m = 1 \,|\, z \,|\, 3d,\ m = 2\rangle$

```
Integrate[Integrate[
  Integrate[- Sqrt[1/a^3] E^(-r/(2 a) - I ϕ) r Sin[θ]/(8 a Sqrt[π]) Sqrt[1/a^3] E^(-r/(3 a) + 2 I ϕ) r^2 Sin[θ]^2/(162 a^2 Sqrt[π]) r Cos[θ], {ϕ, 0, 2 π}] r^2 Sin[θ],
  {θ, 0, π}], {r, 0, ∞}, Assumptions → Re[a] > 0]
```

```
0
```

Therefore,

$W_i = \int \frac{d\vec{q}}{(2\pi\hbar)^3} \frac{2\pi c q}{\hbar} \sum_\lambda \left| -e \frac{165888\, a}{78125} \left(\epsilon_{\lambda x}{}^*(\vec{q}) + i\,\epsilon_{\lambda y}{}^*(\vec{q})\right)\right|^2 2\pi\,\delta(E_f - E_i)$

Using the definition of the polarization vectors, Eqs. (16, 17, 18), we find

$\vec{\epsilon}_+(\vec{q}) = \frac{1}{\sqrt{2}} (\cos\theta\cos\phi - i\sin\phi,\ \cos\theta\sin\phi + i\cos\phi,\ -\sin\theta)$

$\vec{\epsilon}_-(\vec{q}) = \frac{1}{\sqrt{2}} (-\cos\theta\cos\phi - i\sin\phi,\ -\cos\theta\sin\phi + i\cos\phi,\ \sin\theta)$

Therefore

$\epsilon_{+x}{}^*(\vec{q}) + i\,\epsilon_{+y}{}^*(\vec{q}) = \frac{1}{\sqrt{2}} (\cos\theta + 1)\, e^{i\phi}$

$\epsilon_{-x}{}^*(\vec{q}) + i\,\epsilon_{-y}{}^*(\vec{q}) = \frac{1}{\sqrt{2}} (-\cos\theta + 1)\, e^{i\phi}$

These expressions for the amplitudes make sense. When the $m = +2$ state emits a photon with helicity $+1$ and decays into the $m = +1$ state, the angular momentum is not conserved when the photon goes along the negative z-axis, while the conservation is clear along the positive z-axis. Therefore the amplitude has the behavior $1 + \cos\theta$. The same argument goes for a photon with helicity -1.

The decay rate with the positive helicity photon is ($\Delta E = E_{3d} - E_{2p}$)

$\int \frac{d\vec{q}}{(2\pi\hbar)^3} \frac{2\pi c q}{\hbar} \left| -e \frac{165888\, a}{78125} \frac{1}{\sqrt{2}} (\cos\theta + 1)\, e^{i\phi} \right|^2 2\pi\,\delta(E_f - E_i)$

$= \int \frac{q^2\, dq}{(2\pi\hbar)^3} \frac{2\pi c q}{\hbar} \int d\cos\theta \int d\phi\, (\cos\theta + 1)^2 \left| -e \frac{165888\, a}{78125} \frac{1}{\sqrt{2}} \right|^2 2\pi\,\delta(c q - \Delta E)$

$= \int \frac{q^2\, dq}{(2\pi\hbar)^3} \frac{2\pi c q}{\hbar} \frac{8}{3} 2\pi \left| -e \frac{165888\, a}{78125} \frac{1}{\sqrt{2}} \right|^2 2\pi\,\delta(c q - \Delta E)$

$= \frac{q^3 a^2}{\hbar^4} \frac{8}{3} e^2 \frac{13759414272}{6103515625}$

$= \frac{q^3 a^3}{\hbar^3} \frac{8}{3} \frac{e^2}{a} \frac{13759414272}{6103515625} \frac{1}{\hbar}$

Because $\Delta E = \frac{e^2}{2\, 2^2\, a} - \frac{e^2}{2\, 3^2\, a} = \frac{5 e^2}{72 a}$, and $q = \Delta E / c$, and using $\frac{e^2}{\hbar c} = \alpha$, $a = \frac{\hbar^2}{m e^2} = \frac{\hbar c}{\alpha m c^2}$, the decay rate is

$\left(\frac{5 e^2}{72 \hbar c}\right)^3 \frac{8}{3} \frac{e^2}{a} \frac{13759414272}{6103515625} \frac{1}{\hbar}$

$= \frac{98304}{48828125} \alpha^5 \frac{m c^2}{\hbar}$

$= 3.239\ 10^7$ sec

The contribution of the negative helicity photon is exactly the same. Therefore, the total decay rate of $3d \to 2p$ is $0.648\ 10^8\ \text{sec}^{-1}$.

(c)

The result in part (b) is in excellent agreement with what is quoted on the NIST web site, $0.6465\ 10^8\ \mathrm{sec}^{-1}$ for the $3\,d_{5/2} \to 2\,p_{3/2}$ transition. However, it does not agree with $0.5388\ 10^8\ \mathrm{sec}^{-1}$ for the $3\,d_{3/2} \to 2\,p_{1/2}$ transition.

In the above calculation, we completely ignored the electron spin. Because the decay process by the E1 transition does not involve the spin at all, the above result should correctly be the decay rate of the $3\,d$ state. The only complication is that the spin-orbit interaction splits the $3\,d_{5/2}$ and $3\,d_{3/2}$ configurations in the initial state, and $2\,p_{3/2}$ and $2\,p_{1/2}$ configurations in the final state. Because the E1 transition emits a photon with $J = 1$, the $3\,d_{5/2}$ state decays only to $2\,p_{3/2}$, while the $3\,d_{3/2}$ can decay both to $2\,p_{3/2}$ and $2\,p_{1/2}$. Therefore, what we calculated in the part (b) is the total decay rate of the $3\,d_{5/2}$ state, which is the same as the decay rate $3\,d_{5/2} \to 3\,p_{3/2}$, or the total decay rate of $3\,d_{3/2}$, namely the *sum* of the decay rates $3\,d_{3/2} \to 2\,p_{3/2}$ *and* $3\,d_{3/2} \to 2\,p_{1/2}$.

Below, we calculate the relative size of these various decay rates.

For the decay of $3\,d_{5/2}$ to $2\,p_{3/2}$,
$\left|\frac{5}{2}, \frac{5}{2}\right\rangle = |2, 2\rangle \left|\frac{1}{2}, \frac{1}{2}\right\rangle$
$\left|\frac{3}{2}, \frac{3}{2}\right\rangle = |1, 1\rangle \left|\frac{1}{2}, \frac{1}{2}\right\rangle$
Therefore, the dipole transition amplitude is proportional to
$\left\langle \frac{3}{2}, \frac{3}{2} \right| \vec{x} \left| \frac{5}{2}, \frac{5}{2} \right\rangle = \left\langle 1, 1 \right| \vec{x} \left| 2, 2 \right\rangle \propto$ ClebschGordan[{2, 2}, {1, −1}, {1, 1}]
using the Wigner-Eckart theorem. Note that only the tensor operator of $q = 1$, $k = -1$ contributes.

```
ClebschGordan[{2, 2}, {1, -1}, {1, 1}]
```

$$\sqrt{\frac{3}{5}}$$

For the decay of $3\,d_{3/2}$ to $2\,p_{1/2}$, we first work out the composition of the $3\,d_{3/2}$ state,

```
ClebschGordan[{2, 2}, {1/2, -1/2}, {3/2, 3/2}]
```

$$\frac{2}{\sqrt{5}}$$

```
ClebschGordan[{2, 1}, {1/2, 1/2}, {3/2, 3/2}]
```

$$-\frac{1}{\sqrt{5}}$$

$\left|\frac{3}{2}, \frac{3}{2}\right\rangle = \frac{2}{\sqrt{5}} \left|2, 2\right\rangle \left|\frac{1}{2}, \frac{-1}{2}\right\rangle - \frac{1}{\sqrt{5}} \left|2, 1\right\rangle \left|\frac{1}{2}, \frac{1}{2}\right\rangle$

The composition of the $2\,p_{1/2}$ state is

```
ClebschGordan[{1/2, -1/2}, {1, 1}, {1/2, 1/2}]
```

$$-\sqrt{\frac{2}{3}}$$

```
ClebschGordan[{1/2, 1/2}, {1, 0}, {1/2, 1/2}]
```

$\frac{1}{\sqrt{3}}$

$\left|\frac{1}{2}, \frac{1}{2}\right\rangle = -\sqrt{\frac{2}{3}}\left|1, 1\right\rangle\left|\frac{1}{2}, \frac{-1}{2}\right\rangle + \frac{1}{\sqrt{3}}\left|1, 0\right\rangle\left|\frac{1}{2}, \frac{1}{2}\right\rangle$

Therefore, the dipole transition amplitude is proportional to

$$\left(-\sqrt{\frac{2}{3}}\left\langle 1, 1\right|\left\langle\frac{1}{2}, \frac{-1}{2}\right| + \frac{1}{\sqrt{3}}\left\langle 1, 0\right|\left\langle\frac{1}{2}, \frac{1}{2}\right|\right)\vec{x}$$

$$\left(\frac{2}{\sqrt{5}}\left|2, 2\right\rangle\left|\frac{1}{2}, \frac{-1}{2}\right\rangle - \frac{1}{\sqrt{5}}\left|2, 1\right\rangle\left|\frac{1}{2}, \frac{1}{2}\right\rangle\right)$$

$$= -\frac{2\sqrt{2}}{\sqrt{5}}\left\langle 1, 1\right|\vec{x}\left|2, 2\right\rangle - \frac{1}{\sqrt{15}}\left\langle 1, 0\right|\vec{x}\left|2, 1\right\rangle$$

$$\propto -\frac{2\sqrt{2}}{\sqrt{5}}\,\text{ClebschGordan}[\{2, 2\}, \{1, -1\}, \{1, 1\}] - \frac{1}{\sqrt{15}}\,\text{ClebschGordan}[\{2, 1\}, \{1, -1\}, \{1, 0\}]$$

```
ClebschGordan[{2, 1}, {1, -1}, {1, 0}]
```

$\sqrt{\frac{3}{10}}$

```
Simplify[Together[2 √2/√15 √(3/5) + 1/√15 √(3/10)]]
```

$\frac{1}{\sqrt{2}}$

Therefore, the amplitude for $d_{3/2} \to p_{1/2}$ is proportional to $\frac{1}{\sqrt{2}}$, while that for $d_{5/2} \to p_{3/2}$ to $\sqrt{\frac{3}{5}}$, and hence the rates differ by a factor of $\frac{5}{6}$.

For $3\,d \to 2\,p$, the data show 0.6465 (0.5388) sec^{-1} for $3\,d_{5/2} \to 2\,p_{3/2}$ $(3\,d_{3/2} \to 2\,p_{1/2})$. It agrees with the calculated ratio of $\frac{5}{6}$,

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0.6465 * 5/6
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0.53875
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(One can verify the same ratio for $4\,d \to 2\,p$. The data show 0.2063 (0.1719) sec^{-1} for $3\,d_{5/2} \to 2\,p_{3/2}$ $(3\,d_{3/2} \to 2\,p_{1/2})$. It agrees with the calculated ratio of $\frac{5}{6}$)

Therefore, the total lifetime of the $3\,d$ state is given by the calculation in the part (b), yet it is completely consistent with the decay rate for the $3\,d_{3/2} \to 2\,p_{1/2}$ transition listed on the NIST web site.

2. Thomson scattering cross section

(a)

We rely on the amplitude Eq. (80) at the second order in time-dependent perturbation theory.

The first term is given by Eq. (81), where the initial state $|A\rangle = |\vec{p}_i\rangle$ and the final state $|B\rangle = |\vec{p}_f\rangle$ are plane wave states. Note that the box normalization demands $\langle \vec{x} | \vec{p} \rangle = L^{-3/2} e^{i \vec{p}\cdot\vec{x}/\hbar}$. Therefore, using Eq. (81), we find

$$\left\langle B; \vec{q}_f, \lambda_f \left| \frac{e^2}{c^2} \frac{\vec{A}(\vec{x})\cdot\vec{A}(\vec{x})}{2m} \right| A; \vec{q}_i, \lambda_i \right\rangle = \frac{e^2}{m c^2} \frac{2\pi\hbar c^2}{L^3} \frac{1}{\sqrt{\omega_i \omega_f}} \vec{\epsilon}_i \cdot \vec{\epsilon}_f^{\,*} \left\langle B \left| e^{-i(\vec{q}_f - \vec{q}_i)\cdot\vec{x}/\hbar} \right| A \right\rangle$$

$$= \frac{e^2}{m c^2} \frac{2\pi\hbar c^2}{L^3} \frac{1}{\sqrt{\omega_i \omega_f}} \vec{\epsilon}_i \cdot \vec{\epsilon}_f^{\,*} \int d\vec{x} \frac{1}{L^{3/2}} e^{-i \vec{p}_f\cdot\vec{x}/\hbar} e^{-i(\vec{q}_f - \vec{q}_i)\cdot\vec{x}/\hbar} \frac{1}{L^{3/2}} e^{i \vec{p}_i\cdot\vec{x}/\hbar}$$

$$= r_0 \frac{2\pi\hbar c^2}{L^3} \frac{1}{\sqrt{\omega_i \omega_f}} \vec{\epsilon}_i \cdot \vec{\epsilon}_f^{\,*} \frac{1}{L^3} (2\pi\hbar)^3 \delta(\vec{p}_f + \vec{q}_f - \vec{p}_i - \vec{q}_i)$$

The second term has two pieces depending on which vector potential (at the second order) is the annihilation or the creation operator. We need to sum Eq. (83) and Eq. (85), with a careful attention not to use the dipole approximation.

Eq. (83) has the piece $\langle I | \vec{p}\cdot\vec{\epsilon}_i e^{i \vec{q}_i\cdot\vec{x}/\hbar} | A \rangle$ where $\vec{p}, \vec{x}$ are operators, while other quantities are all numbers. The point is that, in Coulomb gauge, $\vec{q}_i \cdot \vec{\epsilon}_i = 0$, and hence $[\vec{p}\cdot\vec{\epsilon}_i, e^{i\vec{q}_i\cdot\vec{x}/\hbar}] = \vec{q}_i \cdot \vec{\epsilon}_i = 0$. Then we can let the momentum operator act directly on the initial state $\vec{p}|A\rangle = \vec{p}_i |A\rangle$,

$$\frac{e^2}{m^2 c^2} \frac{2\pi\hbar c^2}{L^3} \frac{1}{\sqrt{\omega_i \omega_f}} (\vec{p}_f \cdot \vec{\epsilon}_f^{\,*})(\vec{p}_i \cdot \vec{\epsilon}_i) \sum_I \frac{1}{E_i - E_m + i\epsilon} \left\langle B \left| e^{-i\vec{q}_f\cdot\vec{x}/\hbar} \right| I \right\rangle \left\langle I \left| e^{i\vec{q}_i\cdot\vec{x}/\hbar} \right| A \right\rangle$$

$$= r_0 \frac{2\pi\hbar c^2}{L^3} \frac{1}{\sqrt{\omega_i \omega_f}} \frac{(\vec{p}_f\cdot\vec{\epsilon}_f^{\,*})(\vec{p}_i\cdot\vec{\epsilon}_i)}{m} \int \frac{L^3 d\vec{k}}{(2\pi\hbar)^3} \frac{1}{E_i - E_m + i\epsilon} \left\langle \vec{p}_f \left| e^{-i\vec{q}_f\cdot\vec{x}/\hbar} \right| \vec{k} \right\rangle \left\langle \vec{k} \left| e^{i\vec{q}_i\cdot\vec{x}/\hbar} \right| \vec{p}_i \right\rangle$$

$$= r_0 \frac{2\pi\hbar c^2}{L^3} \frac{1}{\sqrt{\omega_i \omega_f}} \frac{(\vec{p}_f\cdot\vec{\epsilon}_f^{\,*})(\vec{p}_i\cdot\vec{\epsilon}_i)}{m} \int \frac{L^3 d\vec{k}}{(2\pi\hbar)^3} \frac{1}{E_i - E_m + i\epsilon} \int d\vec{x} \frac{e^{i(\vec{k} - \vec{p}_f - \vec{q}_f)\cdot\vec{x}/\hbar}}{L^3} \int d\vec{y} \frac{e^{i(\vec{p}_i - \vec{k} + \vec{q}_i)\cdot\vec{y}/\hbar}}{L^3}$$

$$= r_0 \frac{2\pi\hbar c^2}{L^3} \frac{1}{\sqrt{\omega_i \omega_f}} \frac{(\vec{p}_f\cdot\vec{\epsilon}_f^{\,*})(\vec{p}_i\cdot\vec{\epsilon}_i)}{m} \int \frac{L^3 d\vec{k}}{(2\pi\hbar)^3} \frac{1}{E_i - E_m + i\epsilon} \frac{(2\pi\hbar)^3 \delta(\vec{k} - \vec{p}_f - \vec{q}_f)}{L^3} \frac{(2\pi\hbar)^3 \delta(\vec{p}_i - \vec{k} + \vec{q}_i)}{L^3}$$

$$= r_0 \frac{2\pi\hbar c^2}{L^3} \frac{1}{\sqrt{\omega_i \omega_f}} \frac{(\vec{p}_f\cdot\vec{\epsilon}_f^{\,*})(\vec{p}_i\cdot\vec{\epsilon}_i)}{m} \frac{1}{c q_i} \frac{(2\pi\hbar)^3 \delta(\vec{p}_i + \vec{q}_i - \vec{p}_f - \vec{q}_f)}{L^3}$$

The energy denominator is $E_i - E_m = \left(\frac{\vec{p}_i^{\,2}}{2m} + c q_i\right) - \frac{(\vec{p}_i + \vec{q}_i)^2}{2m} \simeq c q_i$ within the non-relativistic approximation $c q,\ c p \ll m c^2$.

The same can be done in Eq. (85) (after recovering the exponential factor),

$$\frac{e^2}{m^2 c^2} \frac{2\pi\hbar c^2}{L^3} \frac{1}{\sqrt{\omega_i \omega_f}} (\vec{p}_f \cdot \vec{\epsilon}_i)(\vec{p}_i \cdot \vec{\epsilon}_f^{\,*}) \int \frac{L^3 d\vec{k}}{(2\pi\hbar)^3} \frac{1}{E_i - E_m + i\epsilon} \left\langle \vec{p}_f \left| e^{i\vec{q}_i\cdot\vec{x}/\hbar} \right| \vec{k} \right\rangle \left\langle \vec{k} \left| e^{-i\vec{q}_f\cdot\vec{x}/\hbar} \right| \vec{p}_i \right\rangle$$

$$= r_0 \frac{2\pi\hbar c^2}{L^3} \frac{1}{\sqrt{\omega_i \omega_f}} \frac{(\vec{p}_f\cdot\vec{\epsilon}_i)(\vec{p}_i\cdot\vec{\epsilon}_f^{\,*})}{m} \frac{1}{-c q_f} \frac{(2\pi\hbar)^3 \delta(\vec{p}_i + \vec{q}_i - \vec{p}_f - \vec{q}_f)}{L^3}$$

The energy denominator in this case is $E_i - E_m = \left(\frac{\vec{p}_i^{\,2}}{2m} + c q_i\right) - \left(\frac{(\vec{p}_i - \vec{q}_f)^2}{2m} + c q_i + c q_f\right) \simeq -c q_f$ within the non-relativistic approximation $c q,\ c p \ll m c^2$.

The transition amplitude is (using Eq. (80))

$$\langle B;\ \vec{q}_f,\ \lambda_f \mid U_I(t_f,\ t_i) \mid A;\ \vec{q}_i,\ \lambda_i\rangle$$
$$= -i\,2\,\pi\,\delta(E_i - E_f)\,r_0\,\frac{2\,\pi\,\hbar\,c^2}{L^3}\,\frac{1}{\sqrt{\omega_i\,\omega_f}}\,\frac{1}{L^3}\,(2\,\pi\,\hbar)^3\,\delta(\vec{p}_f + \vec{q}_f - \vec{p}_i - \vec{q}_i)\left(\vec{\epsilon}_i \cdot \vec{\epsilon}_f^{\;*} + \frac{(\vec{p}_f \cdot \vec{\epsilon}_f^{\;*})(\vec{p}_i \cdot \vec{\epsilon}_i)}{m\,c\,q_i} - \frac{(\vec{p}_f \cdot \vec{\epsilon}_i)(\vec{p}_i \cdot \vec{\epsilon}_f^{\;*})}{m\,c\,q_f}\right)$$

(b)

Now we specialize the amplitude to an electron initially at rest $\vec{p}_i = 0$. It removes both of the second-order pieces, and the amplitude simplifies dramatically to

$$\langle B;\ \vec{q}_f,\ \lambda_f \mid U_I(t_f,\ t_i) \mid A;\ \vec{q}_i,\ \lambda_i\rangle$$
$$= -i\,2\,\pi\,\delta(E_i - E_f)\,r_0\,\frac{2\,\pi\,\hbar\,c^2}{L^3}\,\frac{1}{\sqrt{\omega_i\,\omega_f}}\,\frac{1}{L^3}\,(2\,\pi\,\hbar)^3\,\delta(\vec{p}_f + \vec{q}_f - \vec{p}_i - \vec{q}_i)\,(\vec{\epsilon}_i \cdot \vec{\epsilon}_f^{\;*})$$

This is nothing but Eq. (1) in the exam.

(c)

The absolute square of the amplitude above is

$$\left|\langle B;\ \vec{q}_f,\lambda_f\,|\,U_I(t_f,t_i)\,|\,A;\vec{q}_i,\lambda_i\rangle\right|^2$$

$$= 2\pi\,\delta\,(E_i-E_f)\,\frac{T}{\hbar}\left|r_0\ \frac{2\pi\hbar c^2}{L^3}\ \frac{1}{\sqrt{\omega_i\,\omega_f}}\ \vec{\epsilon}_i\cdot\vec{\epsilon}_f^{\ *}\right|^2\frac{1}{L^3}\,(2\pi\hbar)^3\,\delta\left(\vec{p}_f+\vec{q}_f-\vec{p}_i-\vec{q}_i\right)$$

where I used the replacement $2\pi\,\delta(E_i-E_f)=T/\hbar$, $(2\pi\hbar)^3\,\delta\left(\vec{p}_f+\vec{q}_f-\vec{p}_i-\vec{q}_i\right)=L^3$ for the second power of the delta function following the usual trick. To find the transition probability per unit time, we simply remove a factor of T. To find the cross section, we sum over all final states, namely the states of the final electron and photon, and divide it by the flux of the photon, c/L^3. Maybe a less obvious point is that we also need to sum over the final photon helicities because we don't know in which helicity state the photon will come out after the scattering.

$$\sigma=\frac{L^3}{c}\sum_{\lambda_f}\int\frac{L^3\,d\vec{p}_f}{(2\pi\hbar)^3}\int\frac{L^3\,d\vec{q}_f}{(2\pi\hbar)^3}\,2\pi\,\delta(E_i-E_f)\,\frac{1}{\hbar}\left|r_0\ \frac{2\pi\hbar c^2}{L^3}\ \frac{1}{\sqrt{\omega_i\,\omega_f}}\ \vec{\epsilon}_i\cdot\vec{\epsilon}_f^{\ *}\right|^2\frac{1}{L^3}\,(2\pi\hbar)^3\,\delta\left(\vec{p}_f+\vec{q}_f-\vec{p}_i-\vec{q}_i\right)$$

$$=r_0^{\ 2}\sum_{\lambda_f}\int d\,\vec{q}_f\ \delta(E_i-E_f)\,\frac{c^3}{\hbar^2}\,\frac{1}{\omega_i\,\omega_f}\left|\vec{\epsilon}_i\cdot\vec{\epsilon}_f^{\ *}\right|^2$$

As expected, the fictitious dependence on the size of the universe disappeared.

For the initial state photon, we can pick the initial momentum to be along the positive z-axis $\vec{q}_i=(0,0,q_i)$ without a loss of generality because the electron is at rest. We can pick one helicity arbitrarily because the cross section is the same for either helicity (which follows from the parity invariance of the electromagnetism). Let us pick the positive helicity, $\vec{\epsilon}_i=\frac{1}{\sqrt{2}}\,(1,\ i,\ 0)$. For the final state photon, $\vec{q}_f=q_f(\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta)$, and the helicity can be either positive

$\vec{\epsilon}_+=\frac{1}{\sqrt{2}}\,(\cos\theta\cos\phi-i\sin\phi,\ \cos\theta\sin\phi+i\cos\phi,\ -\sin\theta)$, $\vec{\epsilon}_i\cdot\vec{\epsilon}_f^{\ *}=\frac{1}{2}\,e^{i\phi}(\cos\theta+1)$

or negative

$\vec{\epsilon}_-=\frac{1}{\sqrt{2}}\,(-\cos\theta\cos\phi-i\sin\phi,\ -\cos\theta\sin\phi+i\cos\phi,\ \sin\theta)$, $\vec{\epsilon}_i\cdot\vec{\epsilon}_f^{\ *}=\frac{1}{2}\,e^{i\phi}(-\cos\theta+1)$.

The polar angle dependence makes sense because of the angular momentum conservation, since the electron spin is not playing any roles in these calculations. (If you want to go to the relativistic theory, the electron spin does play a role because of the magnetic moment coupling.) Finally, within the non-relativistic approximation, $q_i=q_f=q$, $\omega_i=\omega_f=c\,q/\hbar$, and hence

$$\sigma=r_0^{\ 2}\int q^2\,d\,q\,d\,\Omega\,\delta(c\,q_f-c\,q_i)\,\frac{c^3}{\hbar^2}\left(\frac{\hbar}{c\,q}\right)^2\frac{1+\cos^2\theta}{2}$$

$$=r_0^{\ 2}\,2\pi\int d\cos\theta\,\frac{1+\cos^2\theta}{2}$$

$$=\frac{8\pi}{3}\,r_0^{\ 2}$$

This is precisely the Thomson cross section calculated in the classical electromagnetism. See, e.g, http://scienceworld.wolfram.com/physics/ThomsonCrossSection.html. Similarly to the Rutherford scattering, the classical calculation and the quantum calculation agree; nature had been kind to us back in the 19th century. Note also the angular dependence $1+\cos^2\theta$ is characteristic to the dipole radiation in classical electromagnetism.

(d)

Let us pretend that the Sun is a sphere of uniform density (which it is not). The solar radius is $R_\odot = 6.961\ 10^{10}$ cm, while its mass is $M_\odot = 1.988\ 10^{33}$ g. The Sun is mostly hydrogen (~75%) and helium (~25%). Because we do only a rough estimate, we assume all hydrogen. Then the number of electrons in the Sun is $N_e = M_\odot / m_p = 1.988\ 10^{33} / 1.673\ 10^{-24} = 1.188\ 10^{57} \sim 1\ 10^{57}$, and its number density $n_e = N_e / (\frac{4\pi}{3} R_\odot{}^2) \sim 7\ 10^{23}\ \text{cm}^{-3}$. (Of course it is much more dense at the core, about $6\ 10^{25}\ \text{cm}^{-3}$.)

The Thomson cross section is $\sigma = 0.665\ 10^{-24}\ \text{cm}^2$. (This is probably why the unit "barn" was chosen to be $10^{-24}\ \text{cm}^2$.) Therefore, the mean free path of the photon is $l = 1/(n_e\,\sigma) \sim 2$ cm, which takes about $t \sim 7\ 10^{-11}$ sec. The number of steps it takes for the photon to diffuse out of the Sun by a random walk is $(R_\odot / l)^2 \sim 1\ 10^{21}$, taking total amount of time of $7\ 10^{10}$ sec ~ 2000 years.

In practice, the Sun's core is much more dense and it takes much longer than this; in any case many thousands of years.

See, e.g., http://www.astronomycafe.net/qadir/ask/a11354.html

Here is an interesting anecdote. When Ray Davis, 2002 Nobel Laureate, discovered that there are only about a third neutrinos from the Sun as what was predicted theoretically by John Bahcall, some people argued that the Sun was shutting off. With neutrinos, we see the status of the Sun's core just 8 minutes ago. With light, we determine the status of the Sun's health thousands of years ago. If the Sun was really shutting off, it could have explained the discrepancy. Fortunately, that wasn't the resolution to the puzzle! It was because the electron neutrinos produced in the nuclear fusion process oscillated into other neutrino species which are not detected in Davis' experiment.

3. Nuclear Magnetic Moment

The magnetic moment operator is $\vec{\mu} = \mu_N\left(Q\,\vec{l} + g\,\vec{s}\right)$, where $Q = +1$ for the proton and $Q = 0$ for the neutron, while $g_p = 5.59$ and $g_n = -3.83$.

The size of the magnetic moment is the expectation value of μ_z in the top state $|\,j\,j\rangle$ where j is the nuclear spin.

In a shell-model orbital with a definite l and $j = l \pm \frac{1}{2}$, we know the Clebsch-Gordan coefficients explicitly

```
Simplify[ClebschGordan[{l, l}, {1/2, 1/2}, {l + 1/2, l + 1/2}]]
```

$(-1)^{4l}$

```
Simplify[ClebschGordan[{l, l}, {1/2, -1/2}, {l - 1/2, l - 1/2}]]
```

$\frac{\sqrt{2}\,\sqrt{l}}{\sqrt{1+2\,l}}$

```
Simplify[ClebschGordan[{l, l - 1}, {1/2, 1/2}, {l - 1/2, l - 1/2}]]
```

$-\frac{1}{\sqrt{1+2\,l}}$

Therefore,
$|\,l+\frac{1}{2},\,l+\frac{1}{2}\rangle = |\,l,\,l\rangle\,|\,\frac{1}{2},\,\frac{1}{2}\rangle$
$|\,l-\frac{1}{2},\,l-\frac{1}{2}\rangle = \sqrt{\frac{2\,l}{2\,l+1}}\,\left|l,\,l\right\rangle\left|\frac{1}{2},\,\frac{-1}{2}\right\rangle - \frac{1}{\sqrt{2\,l+1}}\left|l,\,l-1\right\rangle\left|\frac{1}{2},\,\frac{1}{2}\right\rangle.$

It is now easy to calculate the expectation values
$\langle l+\frac{1}{2},\,l+\frac{1}{2}\,|\,l_z\,|\,l+\frac{1}{2},\,l+\frac{1}{2}\rangle = l$
$\langle l+\frac{1}{2},\,l+\frac{1}{2}\,|\,s_z\,|\,l+\frac{1}{2},\,l+\frac{1}{2}\rangle = \frac{1}{2}$

$\langle l-\frac{1}{2},\,l-\frac{1}{2}\,|\,l_z\,|\,l-\frac{1}{2},\,l-\frac{1}{2}\rangle = \frac{2\,l}{2\,l+1}\,l + \frac{1}{2\,l+1}\,(l-1) = l - \frac{1}{2\,l+1}$
$\langle l-\frac{1}{2},\,l-\frac{1}{2}\,|\,s_z\,|\,l+\frac{1}{2},\,l-\frac{1}{2}\rangle = \frac{2\,l}{2\,l+1}\,\frac{-1}{2} + \frac{1}{2\,l+1}\,\frac{1}{2} = -\frac{2\,l-1}{2\,l+1}\,\frac{1}{2}$

The exectation value of the magnetic moment operator is therefore
$\langle j,\ j\,|\,\mu_z\,|\,j,\ j\rangle = \mu_N(Q\,l + g\,\frac{1}{2})$ for $j = l + \frac{1}{2}$ and
$\langle j,\ j\,|\,\mu_z\,|\,j,\ j\rangle = \mu_N(Q(l - \frac{1}{2\,l+1}) - g\,\frac{2\,l-1}{2\,l+1}\,\frac{1}{2})$ for $j = l - \frac{1}{2}$.

For a hole, the magnetic moment operator flips it sign, while the angular momentum does, too. In the end, the formula remains the same for a hole.

For ^{209}Pb, Fig. 13 in the lecture notes shows a neutron in the $2\,g_{9/2}$ orbital ($l = 4$, even parity) in addition to the doubly-magic ^{208}Pb. Checking the Adopted Level Scheme Diagram (http://ie.lbl.gov/TOI2003/LadderSearch.asp), the ground state is indeed $\frac{9}{2}^+$. Therefore, we expect the magnetic moment to be $\mu_N(0*4 - 3.83*\frac{1}{2}) = -1.91\,\mu_N$. This is to be compared to $-1.44\,\mu_N$, a 25% error.

For ^{209}Bi, Fig. 13 suggests a proton in the $1\,h_{9/2}$ orbital ($l = 5$, odd parity) in addition to the doubly-magic ^{208}Pb. The ground state is indeed $\frac{9}{2}^-$. Therefore, we expect the magnetic moment to be $\mu_N(1\,(5 - \frac{1}{2*5+1}) - 5.59\,\frac{2*5-1}{2*5+1}\,\frac{1}{2}) = +2.62\,\mu_N$. This is to be compared to $+4.11\,\mu_N$, a 36% error.

For ^{207}Pb, Fig. 13 suggests a hole of neutron in the $1\,i_{13/2}$ orbital ($l = 6$, even parity) in addition to the doubly-magic ^{208}Pb. The ground state is, however, $\frac{1}{2}^-$. Looking at nearby orbitals, it appears that the hole is in the $3\,p_{1/2}$ orbital instead. This is not too surprising; the gaps between closed shells are big, while the levels within a shell are closely packed and may well change their relative ordering depending on the details. Therefore, we expect the magnetic moment to be $\mu_N(0*(1 - \frac{1}{2*1+1}) + 3.83\,\frac{2*1-1}{2*1+1}\,\frac{1}{2}) = +0.64\,\mu_N$. This is to be compared to $+0.578\,\mu_N$, a 11% error.

Finally for ^{207}Tl, Fig. 13 suggests a proton in the $1\,g_{9/2}$ orbital ($l = 4$, even parity) in addition to the doubly-magic ^{208}Pb. The ground state is, however, $\frac{1}{2}^+$. Looking at nearby orbitals, it appears that the hole is in the $3\,s_{1/2}$ orbital instead. Therefore, we expect the magnetic moment to be $\mu_N(1*0 + 5.59\,\frac{1}{2}) = +2.79\,\mu_N$. This is to be compared to $+1.88\,\mu_N$, a 48% error.

An error of a few tens of percents should be regarded "not bad" in nuclear physics, where the interactions are strong and perturbation theory is not good.