

Notations

The Dirac γ -matrices in the Weyl representation are defined by

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad (1)$$

or with a short-hand notation

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \quad (2)$$

with $\sigma^\mu = (1, \vec{\sigma})$ and $\bar{\sigma}^\mu = (1, -\vec{\sigma})$. They satisfy the Clifford algebra,

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}. \quad (3)$$

The two-component helicity eigenspinors $(\vec{\sigma} \cdot \vec{p})\chi_\pm(\vec{p}) = \pm|\vec{p}|\chi_\pm(\vec{p})$ are defined by

$$\chi_+(\vec{p}) = \begin{pmatrix} \cos \frac{\theta}{2} \\ \sin \frac{\theta}{2} e^{i\phi} \end{pmatrix}, \quad \chi_-(\vec{p}) = \begin{pmatrix} -\sin \frac{\theta}{2} e^{-i\phi} \\ \cos \frac{\theta}{2} \end{pmatrix}, \quad (4)$$

for $\vec{p} = |\vec{p}|(\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$. The solution to the free Dirac equation is given by

$$\psi(x) = \int d\tilde{p} \sum_{\pm} (u_\pm(p)a_\pm(p)e^{-ip \cdot x} + v_\pm(p)b_\pm^\dagger(p)e^{ip \cdot x}), \quad (5)$$

with $\int d\tilde{p} = \int d^3p/(2\pi)^3 2E_p$, and

$$u_+(p) = \begin{pmatrix} \sqrt{E-p} \\ \sqrt{E+p} \end{pmatrix} \chi_+(\vec{p}), \quad (6)$$

$$u_-(p) = \begin{pmatrix} \sqrt{E+p} \\ \sqrt{E-p} \end{pmatrix} \chi_-(\vec{p}), \quad (7)$$

$$v_+(p) = \begin{pmatrix} -\sqrt{E-p} \\ \sqrt{E+p} \end{pmatrix} \chi_+(\vec{p}), \quad (8)$$

$$v_-(p) = \begin{pmatrix} \sqrt{E+p} \\ -\sqrt{E-p} \end{pmatrix} \chi_-(\vec{p}). \quad (9)$$

The mode operators satisfy the anti-commutation relations

$$\{a_i(p), a_j(q)\} = 0, \quad (10)$$

$$\{a_i^\dagger(p), a_j^\dagger(q)\} = 0, \quad (11)$$

$$\{a_i(p), a_j^\dagger(q)\} = (2\pi)^3 2E_p \delta^3(\vec{p} - \vec{q}) \delta_{ij}, \quad (12)$$

$$\{b_i(p), b_j(q)\} = 0, \quad (13)$$

$$\{b_i^\dagger(p), b_j^\dagger(q)\} = 0, \quad (14)$$

$$\{b_i(p), b_j^\dagger(q)\} = (2\pi)^3 2E_p \delta^3(\vec{p} - \vec{q}) \delta_{ij}, \quad (15)$$

with $i, j = \pm$ and create the following states:

helicity	+1/2	-1/2	(16)
particle	a_+^\dagger	a_-^\dagger	
anti-particle	b_-^\dagger	b_+^\dagger	

It is also useful to define a matrix

$$\gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad (17)$$

which anti-commutes with all γ^μ : $\{\gamma^5, \gamma^\mu\} = 0$. In the massless limit $E \rightarrow p$, the solutions satisfy

$$\gamma^5 u_\pm(p) = \pm u_\pm(p), \quad (18)$$

$$\gamma^5 v_\pm(p) = \pm v_\pm(p), \quad (19)$$

and the eigenvalue of γ^5 is called *chirality*. The projection operators $P_\pm = (1 \pm \gamma^5)/2$ single out solutions with a definite chirality.

HW #3, due Sep 20

1. The non-relativistic limit and g -factor. We discuss the non-relativistic limit of a Dirac field in the following steps. First, we employ the Pauli–Dirac representation of the Dirac γ -matrices; *e.g.*, γ^i are the same as in Weyl representation, but

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (1)$$

This representation makes it easier to take the non-relativistic limit. We also would like to include a vector potential so that we can derive the g -factor of a Dirac particle. For this purpose, replace all spatial derivatives $\vec{\nabla}$ by $\vec{\nabla} - ie\vec{A}$.

- (1) Check that the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ is satisfied with the γ^0 matrices in Pauli–Dirac representation.
- (2) Write the Dirac equation using

$$\psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix}, \quad (2)$$

with ϕ and χ both two-component spinors.

- (3) Rewrite the equation using $\psi' = \psi e^{imt}$ and $\chi' = \chi e^{imt}$.
- (4) Assume ϕ' is $O(1)$, and χ' is $O(v) \ll 1$, where $v = p/m$ is the velocity. Obtain an approximate solution for χ' at $O(v)$. (Hint: ϕ' behaves as $\sim e^{-iEt+imt} \sim e^{-i\frac{1}{2}mv^2t}$, and hence $i\partial/\partial t$ is only of $O(mv^2)$.)
- (5) By eliminating χ' using the above solution, obtain the following equation,

$$i\frac{\partial}{\partial t}\phi' = \frac{[\vec{\sigma} \cdot (-i\vec{\nabla} - e\vec{A})]^2}{2m}\phi' + O(mv^3). \quad (3)$$

- (6) Use the identity $\sigma^i\sigma^j = \delta^{ij} + i\epsilon^{ijk}\sigma^k$ and rewrite the equation in the form

$$i\frac{\partial}{\partial t}\phi' = \frac{(-i\vec{\nabla} - e\vec{A})^2}{2m}\phi' - g\frac{e}{2m}\frac{\vec{\sigma}}{2} \cdot \vec{B} + O(mv^3). \quad (4)$$

What is g ?

2. Helicity. We would like to see the helicities of each states $a_{\pm}^{\dagger}(p)|0\rangle$ and $b_{\pm}^{\dagger}(p)|0\rangle$. Following the Noether procedure, we know that there are conserved angular momentum operators,

$$\vec{J} = \int d^3x : \psi^{\dagger} \left(-i\vec{x} \times \vec{\nabla} + \frac{\vec{\sigma}}{2} \right) \psi : \quad (5)$$

Here, $::$ indicates the normal ordering, *i.e.*, the annihilation operators are moved to the right while the creation operators to the left using the anti-commutation relations.

(1) Show that

$$(\vec{J} \cdot \vec{P})a_{\pm}^{\dagger}(p)|0\rangle = \pm \frac{1}{2}|\vec{p}|a_{\pm}^{\dagger}(p)|0\rangle. \quad (6)$$

Here, the momentum operator \vec{P} has an eigenvalue \vec{p} on this state. (Hint: $\vec{x} \times (-i\vec{\nabla})$ part vanishes in the end using various δ -functions and the fact $(\vec{x} \times \vec{p}) \cdot \vec{p} = 0$.)

(2) Show that

$$(\vec{J} \cdot \vec{P})b_{\pm}^{\dagger}(p)|0\rangle = \mp \frac{1}{2}|\vec{p}|b_{\pm}^{\dagger}(p)|0\rangle. \quad (7)$$

- (3) Imagine a particle with a positive helicity as an object spinning around its momentum direction clockwise. Suppose you try to “catch up” with the particle. When you pass the particle by and look “back” at it, what helicity do you observe? Using this example, can you explain why do you need two states with opposite helicities (each for particle and anti-particle)?
- (4) If a Dirac particle is massless, the upper two and lower two components do not couple in the Lagrangian, and hence you can remove a half, say, lower two components. Write down the Lagrangian using only two components. (Such a field is called a Weyl field.)
- (5) Using the free particle solutions to the Dirac equation, show that a Weyl field contains only $u_{-}(p)$ and $v_{-}(p)$ solutions.
- (6) What helicity does a particle or anti-particle state have? Why does it not contradict with the discussion in (3)?