HW #3, due Feb 11

1. The standard basis for the fundamental representation of $SU(3)$ is

$$T^1 = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad T^2 = \frac{1}{2}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad T^3 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$T^4 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad T^5 = \frac{1}{2}\begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix},$$

$$T^6 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad T^7 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \qquad T^8 = \frac{1}{2\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

(a) Explain why there are exactly eight matrices in the basis.

(b) Evaluate some of the commutators of these matrices and check that f^{abc} is totally antisymmetric. (To do it all is probably too much work by hand, but if you have access to Mathematica or Maple, you can do it all easily.)

(c) Check the orthogonality condition $\mathrm{Tr} T^a T^b = C(r)\delta^{ab}$ and evaluate the constant $C(r)$ for this representation.

(d) Compute the quadratic Casimir operator $C_2(r) = T^a T^a$.

(e) Show that for any representation matrices $[T^a, T^b] = i f^{abc} T^c$, the matrices $T'^a \equiv -T^{a*}$ also satisfy the same commutation relations. (These are the representation matrices for anti-quarks.)

(f) Show that both $C(r)$ and $C_2(r)$ are the same for quarks and anti-quarks.

2. In higher orders of perturbation theory, the runnig for the QCD strong coupling constant $\alpha_s = g_s^2/4\pi$ is given by

$$\mu \frac{d\alpha_s}{d\mu} = -\frac{\beta_0}{2\pi}\alpha_s^2 - \frac{\beta_1}{4\pi^2}\alpha_s^3 - \frac{\beta_2}{(64\pi^3}\alpha_s^4 - \cdots, \tag{1}$$

where $\beta_0 = 11 - \frac{2}{3}n_f$, $\beta_1 = 51 - \frac{19}{3}n_f$, and $\beta_2 = 2857 - \frac{5033}{9}n_f + \frac{325}{27}n_f^2$.

(a) Solve this equation numerically with the boundary condition $\alpha_s(m_Z) = 0.119 \pm 0.03$ with $n_f = 5$ down to $m_b = 4.2$ GeV. Use $m_Z = 91.187$ GeV.

(b) Solve this equation further down from m_b to $m_c = 1.3$ GeV using $n_f = 4$.

(c) Solve this equation further down from m_c to the energy scale $\mu = \Lambda$ where α_s diverges, and determine Λ.