HW #3

1. quark asymmetry

We first need to convert the baryon-to-photon ratio to the baryon-to-entropy ratio. The current entropy density comes from both the photons and neutrinos. We take $\hbar = c = k = 1$. For the given temperature, the number density is (for bosons and fermions, respectively)

$$n = g \int \frac{d^3 p}{(2\pi)^3} \frac{1}{e^{\beta p}-1} = g \int \frac{4\pi p^2 \, d p}{(2\pi)^3} \sum_{n=1}^{\infty} e^{-n\beta p} = g \sum_{n=1}^{\infty} \frac{\Gamma(3)}{2\pi^2} \frac{T^3}{n^3} = g \frac{\zeta(3)}{\pi^2} T^3$$

$$n = g \int \frac{d^3 p}{(2\pi)^3} \frac{1}{e^{\beta p}+1} = g \int \frac{4\pi p^2 \, d p}{(2\pi)^3} \sum_{n=1}^{\infty} (-1)^{n-1} e^{-n\beta p} = g \sum_{n=1}^{\infty} \frac{\Gamma(3)}{2\pi^2} \frac{(-1)^{n-1} T^3}{n^3} = g \frac{3}{4} \frac{\zeta(3)}{\pi^2} T^3$$

The energy density is

$$\rho = g \int \frac{d^3 p}{(2\pi)^3} \frac{p}{e^{\beta p}-1} = g \int \frac{4\pi p^3 \, d p}{(2\pi)^3} \sum_{n=1}^{\infty} e^{-n\beta p} = g \sum_{n=1}^{\infty} \frac{\Gamma(4)}{2\pi^2} \frac{T^4}{n^4} = g \frac{3\zeta(4)}{\pi^2} T^4 = g \frac{\pi^2}{30} T^4$$

$$\rho = g \int \frac{d^3 p}{(2\pi)^3} \frac{p}{e^{\beta p}+1} = g \int \frac{4\pi p^3 \, d p}{(2\pi)^3} \sum_{n=1}^{\infty} (-1)^{n-1} e^{-n\beta p} = g \sum_{n=1}^{\infty} \frac{\Gamma(4)}{2\pi^2} \frac{(-1)^{n-1} T^4}{n^4} = g \frac{7}{8} \frac{\pi^2}{30} T^4$$

The entropy density is given by the first law of thermodynamics

$$d U = T \, d S - p \, d V$$

Taking $d V = 0$,

$$d U = d\left(g \frac{\pi^2}{30} T^4 V\right) = g \frac{2\pi^2}{15} T^3 V \, d T = T \, d S = T \, d(s V)$$

and hence $d s = g \frac{2\pi^2}{15} T^2 \, d T$, and $s = g \frac{2\pi^2}{45} T^3$. For fermions, $s = g \frac{7}{8} \frac{2\pi^2}{45} T^3$.

The number and entropy densities of the photons are

$n = 2 \frac{\zeta(3)}{\pi^2} {T_0}^3$ and $s = 2 \frac{2\pi^2}{45} {T_0}^3$.

On the other hand, neutrinos were in equilibrium before the decoupling of electorns and positrons, and hence the entropy was shared by photons, electrons, and positrons back then. Therefore the three generations of neutrinos (ν_e, ν_μ, ν_τ) have less entropy than the photons by

$$s_\nu : s_\gamma = 2 \cdot 3 \cdot \frac{7}{8} : (4 \cdot \frac{7}{8} + 2) = \frac{21}{4} : \frac{11}{2} = \frac{21}{22} : 1.$$

Therefore,

$$s = s_\gamma + s_\nu = (1 + \frac{21}{22}) s_\gamma = \frac{43}{22} \frac{2\pi^2}{45} \frac{\pi^2}{\zeta(3)} n_\gamma.$$

We hence find

$$\frac{n_B}{s} = \frac{n_B}{n_\gamma} \frac{n_\gamma}{s} = \frac{22}{43} \frac{45\zeta(3)}{2\pi^4} 6.5\ 10^{-10} = 9.2\ 10^{-11}$$

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N[22/43 45 Zeta[3]/(2 π^4) 6.5 10^-10]
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9.2337 × 10^-11
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This ratio n_B / s is conserved as long as there is no heat injection to the universe and the baryon number is not violated.

Just above the QCD phase transition, the relativistic species are u, d, s, e, μ, γ, g, ν_e, ν_μ, ν_τ. Remember that quarks come in three colors and gluons in eight. Therefore, the total entropy is $s = g_* \frac{2\pi^2}{45} T^3$ with

$$g_* = (4 \cdot 3 \cdot 3 \cdot \frac{7}{8} + 4 \cdot 2 \cdot \frac{7}{8} + 2 + 2 \cdot 8 + 2 \cdot 3 \cdot \frac{7}{8}) = \frac{247}{4}.$$

Because each quark carries the baryon number $1/3$ (and anti-quark $-1/3$), $n_B = \frac{1}{3}(n_q - n_{\bar{q}})$. On the other hand, total number of quarks *and* anti-quarks is

$$n_q + n_{\bar{q}} = 4 \cdot 3 \cdot 3 \cdot \frac{3}{4} \frac{\zeta(3)}{\pi^2} T^3 = 27 \frac{\zeta(3)}{\pi^2} T^3 = 27 \frac{\zeta(3)}{\pi^2} \frac{1}{g_*} \frac{45}{2\pi^2} s = \frac{1215\zeta(3)}{2\pi^4 g_*} s.$$

Therefore,

$A_q = \frac{n_q - n_{\bar{q}}}{n_q + n_{\bar{q}}} = (3\, n_B) \Big/ \left(\frac{1215\, \zeta(3)}{2\, \pi^4\, g_*}\, s \right) = \frac{6\, \pi^4\, g_*}{1215\, \zeta(3)}\, \frac{n_B}{s} = \frac{6\, \pi^4\, g_*}{1215\, \zeta(3)}\, \frac{22}{43}\, \frac{45\, \zeta(3)}{2\, \pi^4}\, \eta = \frac{22}{387}\, g_*\, \eta = 2.3 \times 10^{-9}$.

$$\frac{22}{387}\,\frac{247}{4}\,6.5\;10^{-9}$$

2.28172×10^{-8}

Therefore, for one billion anti-quarks, there were about one billion and 4.6 quarks. Billon quarks and anti-quarks have annihilated to leave 4.6 quarks behind.

2. Matter-radiation equality

Given the measured current composition of the universe, we can work them backwards to determine when the matter and radiation components were equal in size. This particular epoch is called matter-radiation equality and is important for the study of structure formation. Basically the density fluctuation did not grow before the matter-radiation equality, and it started to grow and eventually led to formation of galaxies and their clusters after the equality.

Because the matter density goes down as R^{-3} and radiation as R^{-4}, their ratio goes as R^{-1}. In other words, if you go back in time, or equivalently, higher redshift $1 + z \propto R^{-1}$, the radiation component becomes more and more important. Therefore, the equality must have occured when $1 + z_{\text{eq}} = \Omega_M / \Omega_R$, where Omegas are the current values. WMAP data says $\Omega_M = 0.27$ (0.23 for dark matter, 0.044 for atoms).

There is one complication which turns out to be not an issue. The energy density of radiation now is due to photons and neutrinos. Because neutrinos were found to have mass, they may not be quite "radiation" any more, and the evolution of their energy density depends on the actual mass. However, as it turns out, the neutrinos were relativistic at the time of the matter-radiation equality and we can pretend they have been relativistic all along. After all, there is no direct measurement of the neutirno energy density now we can base our calculations on. We rely on theory, namely that the neutrinos must have been in equlibrium at MeV temperatures and had dropped out of the equilibrium (as confirmed by the success of the Big-Bang Nucleosynthesis theory), hence their energy density and momentum density must have scaled as radiation down to their "temperature" of the order of their mass. We worked out the "temperature" of the neutrinos in class, and found that the neutrino temperatures were related to the photon temperature by ${T_\nu}^3 = \frac{4}{11}\, {T_\gamma}^3$ after the decoupling of the positrons which changed the effective number of degrees of freedom for the photon-electron gas from $4 \cdot \frac{7}{8} + 2 = \frac{11}{2}$ to 2. Therefore, we pretend that the radiation energy density "now" is given by
$\rho_R = 3 \cdot 2 \cdot \frac{7}{8}\, \frac{\pi^2}{30}\, {T_\nu}^4 + 2\, \frac{\pi^2}{30}\, {T_\gamma}^4$
To recover various constants, we pay attention to the dimensions and find
$\rho_R = (\hbar c)^{-3} \left(3 \cdot 2 \cdot \frac{7}{8}\, \frac{\pi^2}{30}\, (k\, T_\nu)^4 + 2\, \frac{\pi^2}{30}\, (k\, T_\gamma)^4 \right) = 0.438 \text{ eV cm}^{-3}$.

$$(1.973\;10^{-5})^{-3} \left(3\;2\;\frac{7}{8}\;\frac{\pi^2}{30}\;\left(\frac{4}{11}\right)^{4/3}\;(2.725 * 8.617\;10^{-5})^4 + 2\;\frac{\pi^2}{30}\;(2.725 * 8.617\;10^{-5})^4 \right)$$

0.437894

Compared to the critical density $\rho_c = 1.054 \times 10^4\, h^2 \text{ eV cm}^{-3}$, we find $\Omega_R\, h^2 = 4.16 \times 10^{-5}$.

0.438 / (1.054 10^4)

0.000041556

Therefore, $1 + z_{\text{eq}} = \Omega_M / \Omega_R = 6500\, h^2 \approx 3300$ as the redshift at the matter-radiation equality.

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0.27 / (4.16 10^-5)
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6490.38

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% * 0.71^2
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3271.8

3. Photon Decoupling

Photons decouple from the thermal bath when most of the electrons are bound to the atoms and the photon-atom scattering rate becomes smaller than the expansion rate of the universe. A complete calculation must include all energy levels of both hydrogen and helium (other elements from the BBN can be safely ignored and no stellar nucleosynthesis took place before this epoch), but we resort to a simplified model: we consider only the ground state of hydrogen and ignore helium ($Y_P \sim 24\,\%$ in mass fraction). At a given temperature T, the fraction of electrons in the bound state is $1/(1+e^{-E_b/T})$, where $E_b = 13.6\,\text{eV}$. The cross section of a photon on a free electron is the Thomson cross section $\sigma_T = \frac{8\pi}{3}\,\frac{\alpha^2}{{m_e}^2} = 0.665\,\text{barn}$(it turns out that $T \ll E_b \ll m_e\,c^2$ in this epoch and relativistic effects can be ignored), while that on a bound electron is given by the Reyleigh cross section, $\sigma_R \sim \sigma_T\,(E_\gamma / E_b)^4 \ll \sigma_T$. Because the Rayleigh cross section is so much smaller than the Thomson cross section, we ignore it. The number density of the electrons is far smaller than that of the photons, $n_e / n_\gamma = n_p / n_\gamma = (1 - Y_P)\,n_B / n_\gamma = (1 - Y_P)\,\eta$. Among them, only a small fraction $X_e \ll 1$ is still ionized. Therefore the rate of a photon to interact with an electron is
$\Gamma_T = \sigma_T\,n_e^{\text{free}} = \sigma_T\,X_e(1 - Y_P)\,\eta\,n_\gamma$. The ion fraction can be computed with the Boltzmann statistics (because we can ignore Fermi statistics in a dilute gas) based on the chemical equilibrium $H + \gamma \leftrightarrow p + e$. The Boltzmann statistics with $E = m + \frac{p^2}{2m}$ gives

$$n = g \int \frac{d^3 p}{(2\pi)^3}\, e^{-\beta(m+p^2/2m-\mu)} = g(\frac{m T}{2\pi})^{3/2}\, e^{\beta(\mu-m)}$$

$$n_e^{\text{free}} = 2\,(\frac{m_e T}{2\pi})^{3/2}\, e^{\beta(\mu_e - m_e)},$$

$$n_p^{\text{free}} = 2\left(\frac{m_p T}{2\pi}\right)^{3/2}\, e^{\beta(\mu_p - m_p)},$$

$$n_H = 2\,(\frac{m_H T}{2\pi})^{3/2}\, e^{\beta(\mu_H - m_H)},$$

where $m_H = m_e + m_p - E_b$, and the chemical eqeuilibrium requires

$$\mu_e + \mu_p = \mu_H$$

(here we used $n_\gamma = 0$). We find (ignoring the difference between m_p and m_H in the prefactor but retaining it in the exponent)

$$n_H = n_e^{\text{free}}\, n_p^{\text{free}}\, \frac{1}{2}\left(\frac{2\pi}{m_e T}\right)^{3/2} e^{\beta E_b}.$$

Because the charge neutrality requires $n_e^{\text{free}} = n_p^{\text{free}} = X_e\, n_e$, we find

$$(1 - X_e)\,n_e = (X_e\, n_e)^2\, \frac{1}{2}\left(\frac{2\pi}{m_e T}\right)^{3/2} e^{\beta E_b},$$

and hence (using $X_e \ll 1$),

$$X_e = \left(\frac{2}{n_e}\,(\frac{m_e T}{2\pi})^{3/2}\, e^{-\beta E_b}\right)^{1/2}.$$

Hence the rate for a photon to be scattered by an electron is

$$\Gamma_T = \sigma_T\, X_e\, n_e = \sigma_T\left(\frac{2}{n_e}\,(\frac{m_e T}{2\pi})^{3/2}\, e^{-\beta E_b}\right)^{1/2} n_e = \sigma_T\left(2\,\eta\, n_\gamma\,(\frac{m_e T}{2\pi})^{3/2}\, e^{-\beta E_b}\right)^{1/2} = \sigma_T\left(4\,\eta\,\frac{\zeta(3)}{\pi^2}\, T^3\,(\frac{m_e T}{2\pi})^{3/2}\, e^{-\beta E_b}\right)^{1/2}$$

We compare it to the expansion rate of the universe. Because it is already matter dominated but yet dark-energy dominated, the expansion rate is given by

$$H^2 = {H_0}^2\, \Omega_M\,(1+z)^3 = {H_0}^2\, \Omega_M\left(\frac{T}{T_0}\right)^3,$$

where $T_0 = 2.725\,K$. The decoupling temperature is then given by the equation

$${\sigma_T}^2\, 4\,\eta\, \frac{\zeta(3)}{\pi^2}\, T^3\,(\frac{m_e T}{2\pi})^{3/2}\, e^{-\beta E_b} = {H_0}^2\, \Omega_M\left(\frac{T}{T_0}\right)^3$$

Recovering various constants of nature, both sides of the equation have the dimension of $[\text{time}]^{-2}$:

$$c^2\, {\sigma_T}^2\, 4\,\eta\, \frac{\zeta(3)}{\pi^2}\, \frac{1}{(\hbar c)^6}\, (k\,T)^3\left(\frac{m_e\, c^2\, k\,T}{2\pi}\right)^{3/2} e^{-\beta E_b} = {H_0}^2\, \Omega_M\left(\frac{T}{T_0}\right)^3$$

$$\texttt{FindRoot}\left[c^2\,\sigma_T{}^2\,4\,\eta\,\frac{\texttt{Zeta[3]}}{\pi^2}\,\frac{1}{\texttt{hbarc}^6}\,T^3\left(\frac{m_e\,T}{2\,\pi}\right)^{3/2} E^{-\epsilon_b/T} == H_0{}^2\,\Omega_M\left(\frac{T}{T_0}\right)^3 \;/.\; \{c\to 3.00\;10^{10},\ \sigma_T\to 0.665\;10^{-24},\ \eta\to 6.5\;10^{-10},\ m_e\to 0.511\;10^6,\ \texttt{hbarc}\to 0.197\;10^{-4},\ \epsilon_b\to 13.6,\ H_0\to 0.71\;100\,/\,(10^6\;3.086\;10^{13}),\ \Omega_M\to 0.27,\ T_0\to 2.725*8.617\;10^{-5}\},\ \{T,\ 1\}\right]$$

{T → 0.247303}

Therefore, the redshift at the photon decoupling is $1+z_{\text{dec}} \approx 1050$

$$T/T_0 \;/.\; \{T_0\to 2.725*8.617\;10^{-5}\} \;/.\; \%[[1]]$$

1053.19

This is shortly after the "recombination" when most of the electrons became bound. In class, we used $X_e = 0.1$ as the definition of the recombination, and found $1+z_{\text{rec}} \approx 1240$:

$$\texttt{FindRoot}\left[\frac{2}{\eta\,2\,\frac{\texttt{Zeta[3]}}{\pi^2}\,T^3}\left(\frac{m_e\,T}{2\,\pi}\right)^{3/2} E^{-\epsilon_b/T} == 0.1^2 \;/.\; \{\eta\to 6.5\;10^{-10},\ m_e\to 0.511\;10^6,\ \epsilon_b\to 13.6\},\ \{T,\ 1\}\right]$$

{T → 0.291385}

$$T/T_0 \;/.\; \{T_0\to 2.725*8.617\;10^{-5}\} \;/.\; \%[[1]]$$

1240.92

Universe was pretty busy, going through the matter-radiation equality, recombination, and photon decouping within a factor of three of the redshift.