HW #6 (229C), due Dec 16, 4pm

1. Solve the equation of motion for the inflaton field

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0 \tag{1}$$

for the quadratic potential $V(\phi) = \frac{m^2}{2}\phi^2$ together with the Friedmann equation

$$H^2 = \frac{8\pi}{3} G_N \left(\frac{1}{2}\dot{\phi}^2 + V(\phi) \right). \tag{2}$$

   (a) First for the slow-roll regime, $\phi \ll M_{Pl} = 1/\sqrt{8\pi G_N}$, neglect the kinetic energy relative to the potential energy in the Friedmann equation, and neglect $\ddot{\phi}$ in the field equation. Solve the equations analytically. Verify that indeed the kinetic energy and $\ddot{\phi}$ can be ingored relative to other terms when $\phi \gg M_{Pl}$. Also show that the scale factor grows exponentially with time.

   (b) Second for the oscillating regime, show that

$$\phi(t) = \phi(t_1) \left(\frac{t_1}{t} \right) \cos m(t - t_1) \tag{3}$$

   is an approximate solution when $\phi(t_1) \ll M_{Pl}$, $mt \gg 1$, namely that the leading term in $1/(mt)$ cancels, and that the subleading term cancels for a specific choice of t_1. Also show that the universe behaves as matter-dominated one.

   (c) Finally solve the equations numerically for $m = 10^{-2} M_{Pl}$ and the initial conditions $\phi(0) = 100 M_{Pl}$ and $\dot{\phi}(0) = 0$, and show that the solution intepolates between the two regimes.

Optional Take a massless scalar field in the inflating background,

$$S = \int dt d^3\vec{x} \sqrt{-g} \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi. \tag{4}$$

Using the conformally flat metric $ds^2 = R(\eta)^2 (d\eta^2 - d\vec{x}^2)$, where $R(\eta) = (H\eta)^{-1}$, rewrite the action for $\tilde{\phi} = \phi/(H\eta)$ and show that the equation of motion is

$$\ddot{\tilde{\phi}} + \frac{2}{\eta^2}\tilde{\phi} - (\vec{\nabla}\tilde{\phi})^2 = 0. \tag{5}$$

Then using the mode expansion calculate the correlation function $\langle \phi(\eta, \vec{x}) \phi(\eta, \vec{y}) \rangle = (H\eta)^2 \langle \tilde{\phi}(\eta, \vec{x}) \tilde{\phi}(\eta, \vec{y}) \rangle$.