Quantum Fluctuation in the Inflating Universe

This is the optional problem in HW #6.

We start with the action for a massless scalar field

$$S = \int dt d^3\vec{x}\sqrt{-g}\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi. \quad (1)$$

With the RFW metric $ds^2 = dt^2 - R(t)^2 d\vec{x}^2$, the scale factor grows exponentially $R(t) = R(0)e^{Ht}$. We now switch to the conformally flat metric $ds^2 = R(\eta)^2(d\eta^2 - d\vec{x}^2)$. Therefore, $dt = R(\eta)d\eta$, or equivalently, $d\eta = \pm R(t)^{-1}dt = \pm R(0)^{-1}e^{-Ht}dt$. Note that η is nothing but the size of the particle horizon, namely the comoving distance for which the light could propagate since the initial time. This differential equation is easy to integrate, and we find

$$\eta = \eta_0 \mp \frac{1}{H}R(0)^{-1}e^{-Ht} = \eta_0 \mp \frac{1}{H}R(\eta)^{-1}. \quad (2)$$

In class, we chose $\eta_0 = 0$ and $\eta > 0$, then $R(\eta) = R(0)/(H\eta)$ and $t = -\frac{1}{H}\ln(HR(0)\eta)$. With this choice, note that η *decreases* as t increases. Many of you found this confusing. Let me instead take the opposite sign here, $\eta_0 = 0$ with $\eta < 0$, and hence $R(\eta) = -R(0)/(H\eta)$ and $t = -\frac{1}{H}\ln(-\eta HR(0))$. This way, both t and η increase, while η is limited to $\eta < 0$. The action in this metric is

$$S = \int d\eta d^3\vec{x}\frac{1}{(H\eta)^4}\frac{1}{2}(H\eta)^2(\dot{\phi}^2 - (\vec{\nabla}\phi)^2) = \int d\eta d^3\vec{x}\frac{1}{(H\eta)^2}\frac{1}{2}(\dot{\phi}^2 - (\vec{\nabla}\phi)^2). \quad (3)$$

Here and below, the "dot" means η derivative.

Using the new variable $\tilde{\phi} = \phi/(H\eta)$, $\dot{\phi} = H\tilde{\phi} + H\eta\dot{\tilde{\phi}}$, and the action is

$$\begin{aligned} S &= \int d\eta d^3\vec{x}\frac{1}{(H\eta)^2}\frac{1}{2}((H\tilde{\phi} + H\eta\dot{\tilde{\phi}})^2 - (\vec{\nabla}\tilde{\phi})^2) \\ &= \int d\eta d^2\vec{x}\frac{1}{2}\left(\dot{\tilde{\phi}}^2 + \frac{2}{\eta}\tilde{\phi}\dot{\tilde{\phi}} + \frac{1}{\eta^2}\tilde{\phi}^2 - (\vec{\nabla}\tilde{\phi})^2\right). \end{aligned} \quad (4)$$

The second term in the parentheses can be integrated by parts because $2\tilde{\phi}\dot{\tilde{\phi}} = \frac{\partial}{\partial\eta}\tilde{\phi}^2$,

$$S = \int d\eta d^3\vec{x}\frac{1}{(H\eta)^2}\frac{1}{2}((H\tilde{\phi} + H\eta\dot{\tilde{\phi}})^2 - (\vec{\nabla}\tilde{\phi})^2)$$

$$
\begin{aligned}
&= \int d\eta d^3\vec{x}\frac{1}{2}\left(\dot{\tilde{\phi}}^2 + \frac{1}{\eta^2}\tilde{\phi}^2 + \frac{1}{\eta^2}\tilde{\phi}^2 - (\vec{\nabla}\tilde{\phi})^2\right) \\
&= \int d\eta d^3\vec{x}\frac{1}{2}\left(\dot{\tilde{\phi}}^2 + \frac{2}{\eta^2}\tilde{\phi}^2 - (\vec{\nabla}\tilde{\phi})^2\right). \qquad (5)
\end{aligned}
$$

In the infinite past, $\eta \to -\infty$, the action is the same as in the ordinary Minkowski space with this coordinate system. Therefore, the field is expanded as usual

$$
\tilde{\phi}(\eta,\vec{x}) = \int \frac{d^3\vec{p}}{(2\pi)^3 2p}\left(a(\vec{p})e^{-ip\eta+i\vec{p}\cdot\vec{x}} + a^\dagger(\vec{p})e^{ip\eta-i\vec{p}\cdot\vec{x}}\right). \qquad (6)
$$

Here we already used the fact that the field is massless and hence $E_p = p$.

The creation and annihilation operators satisfy the commutation relation

$$
[a(\vec{p}), a^\dagger(\vec{q})] = (2\pi)^3 2p\delta^3(\vec{p}-\vec{q}) \qquad (7)
$$

(see, *e.g.*., Peskin and Schroeder). The ground state is defined by $a(\vec{p})|0\rangle = 0$. Note that we are using the Heisenberg picture where the field operator depends on "time" η.

At finite "time," the Euler–Lagrange equation is

$$
\ddot{\tilde{\phi}} - \frac{2}{\eta^2}\tilde{\phi} - \Delta\tilde{\phi} = 0. \qquad (8)
$$

We solve it with the boundary condition that the solution reduces to that in Minkowski space Eq. (6) in the inifinite past $\eta \to -\infty$.[1] Knowing the following solution,

$$
\begin{aligned}
&\left(\frac{d^2}{d\eta^2} - \frac{2}{\eta^2} + k^2\right)e^{-ik\eta}\left(1-\frac{i}{k\eta}\right) \\
&= \left(-k^2\left(1-\frac{i}{k\eta}\right) - 2ik\frac{i}{k\eta^2} - \frac{2i}{k\eta^3} + \left(-\frac{2}{\eta^2}+k^2\right)\left(1-\frac{i}{k\eta}\right)\right)e^{-ik\eta} \\
&= \left(-k^2 + \frac{ik}{\eta} + \frac{2}{\eta^2} - \frac{2i}{k\eta^3} - \frac{2}{\eta^2} + \frac{2i}{k\eta^3} + k^2 - \frac{ik}{\eta}\right)e^{-ik\eta} = 0, \qquad (9)
\end{aligned}
$$

[1] This is an oversimplification for an inflation that lasted for an infinitely long time. For a finite inflationary period, boundary condition needs to be specified at a finite η, instead of $\eta \to -\infty$. This will come back as an issue later on.

we find

$$\tilde{\phi}(\eta,\vec{x}) = \int \frac{d^3\vec{p}}{(2\pi)^3 2p} \left(a(\vec{p}) e^{-ip\eta+i\vec{p}\cdot\vec{x}} \left(1-\frac{i}{p\eta}\right) + a^\dagger(\vec{p}) e^{ip\eta-i\vec{p}\cdot\vec{x}} \left(1+\frac{i}{p\eta}\right)\right). \tag{10}$$

This is a solution to the Euler–Lagrange equation Eq. (8) and correctly reduces to that for the Minkowski space Eq. (6) in the infinite past $\eta \to -\infty$.

The same-time correlation function is then evaluated easily,

$$\begin{aligned}
&\langle 0|\tilde{\phi}(\eta,\vec{x})\tilde{\phi}(\eta,\vec{y})|0\rangle \\
&= \int \frac{d^3\vec{p}}{(2\pi)^3 2p} \int \frac{d^3\vec{q}}{(2\pi)^3 2q} \langle 0| a(\vec{p}) e^{-ip\eta+i\vec{p}\cdot\vec{x}} \left(1-\frac{i}{p\eta}\right) a^\dagger(\vec{q}) e^{iq\eta-i\vec{q}\cdot\vec{y}} \left(1+\frac{i}{q\eta}\right)|0\rangle \\
&= \int \frac{d^3\vec{p}}{(2\pi)^3 2p} \int \frac{d^3\vec{q}}{(2\pi)^3 2q} e^{-ip\eta+i\vec{p}\cdot\vec{x}} \left(1-\frac{i}{p\eta}\right) e^{iq\eta-i\vec{q}\cdot\vec{y}} \left(1+\frac{i}{q\eta}\right)(2\pi)^3 2p\delta^3(\vec{p}-\vec{q}) \\
&= \int \frac{d^3\vec{p}}{(2\pi)^3 2p} e^{i\vec{p}\cdot(\vec{x}-\vec{y})} \left(1-\frac{i}{p\eta}\right)\left(1+\frac{i}{p\eta}\right) \\
&= \int \frac{d^3\vec{p}}{(2\pi)^3 2p} e^{i\vec{p}\cdot(\vec{x}-\vec{y})} \left(1+\frac{1}{(p\eta)^2}\right).
\end{aligned} \tag{11}$$

Note that the first piece in the parentheses exists also in the Minkowski space and hence represents the usual zero-point fluctuation of the field. The second piece is specific to the inflationary cosmology and grows with time $\eta \to 0$.

Going back to the original normalization of the field operator $\phi = (H\eta)\tilde{\phi}$,

$$\begin{aligned}
\langle 0|\phi(t,\vec{x})\phi(t,\vec{y})|0\rangle &= (H\eta)^2 \int \frac{d^3\vec{p}}{(2\pi)^3 2p} e^{i\vec{p}\cdot(\vec{x}-\vec{y})} \left(1+\frac{1}{(p\eta)^2}\right) \\
&= \int \frac{d^3\vec{p}}{(2\pi)^3 2p} e^{i\vec{p}\cdot(\vec{x}-\vec{y})} \left(e^{-2Ht}+\frac{H^2}{p^2}\right)
\end{aligned} \tag{12}$$

Remember that $|\eta|$ is the horizon size. The modes $\lambda\!\!\!^{-} = \hbar/p > H^{-1}e^{-Ht}$ have the wevelengths larger than the horizon and are called the *superhorizon* modes. As time goes on $\eta \to 0$, modes that used to be subhorizon go superhorizon and and the first term in the parentheses can be ignored.

$$\langle 0|\phi(t,\vec{x})\phi(t,\vec{y})|0\rangle = \int_{p<He^{Ht}} \frac{d^3\vec{p}}{(2\pi)^3} e^{i\vec{p}\cdot(\vec{x}-\vec{y})} \frac{H^2}{2p^3}. \tag{13}$$

This is a fascinating result. There is now superhorizon-sized correlation of the field, in apparent violation of causality. Of course, it does not really violate causality because the mode *used to be* subhorizon size and the points x and y were in causal contact back then. However if we see such a correlation in the sky *now*, it appears that there is correlation between parts of the sky without apparent prior causal contact.

This expression cannot be completely right, because it is infrared divergent $p \to 0$. This is an artefact of our oversimplified analysis assuming the Minkowski space only back when $\eta \to -\infty$, namely that the inflation has been going on *since the beginning of the time*. Realistically, we should assume the inflation lasted only over a finite time, and the modes which exited the horizon before the start of the inflation $t = 0$ ($\eta = 1/HR(0)$) should not have received a large fluctuation. Therefore, the integration region for the momentum is rather $H < p < He^{Ht}$. (In practice, this is never really an issue because the modes generated at the beginning of the inflation are presumably still superhorizon size and we have not seen them.)

To see contributions of various modes, let us look at the variance ($\vec{x} = \vec{y}$)

$$\langle \phi^2 \rangle = H^2 \int_H^{He^{Ht}} \frac{4\pi p^2 dp}{(2\pi)^3} \frac{1}{2p^3} = \frac{H^2}{(2\pi)^2} \int_H^{He^{Ht}} \frac{dp}{p} = \frac{H^3}{(2\pi)^2} t. \tag{14}$$

The variance increases as time, reminiscent of a random walk or diffusion process. One can interpret this behavior as follows. During inflation, the volume within H^{-3} is causally connected (horizon) and brews quantum fluctuation. When a mode exists this volume, it loses quantum coherence because it is no longer causally connected, and turns *classical*. Namely that the universe *observes* the mode and the wave function collapses with a definite size and sign. As each mode exists the horizon, it adds to the variance randomly with an equal weight $\frac{dp}{p}$. Hence the total variance $\langle \phi^2 \rangle$ is the sum which grows as a random walk. Fore more discussions on this point, see, *e.g.*, Andrei D. Linde, "Particle Physics and Inflationary Cosmology," Harwood Academic Publishers, 1990. The point that all modes contribute equally as dp/p is the reason why the predicted spectrum is called *scale-invariant*. The fluctuation is also *Gaussian* because the action is purely quadratic and the correlation function is saturated by the two-point ones thanks to the Wick's theorem (again, see Peskin and Schoeder). In sum, scale-invariant Gaussian fluctuation that is apparently acausal is the prediction of inflation.

Once the correlation function is regarded classical, it is used to obtain the prediction on the density fluctuation in inflationary cosmology(see, Scott Dodelson, "Modern Cosmology," Academic Press, 2003). We simply apply the above calculation to the inflaton itself. We used the fact that the quantity

$$\zeta = \frac{\delta\rho}{\rho + p} \tag{15}$$

is conserved throughout the superhorizon evolution. During the inflation $\rho + p = \dot{\phi}^2$ while $\delta\rho = V'(\phi)\delta\phi$. The field ϕ is slow-rolling down the potential classically, and the correlation function calculated above applies to its fluctuation $\delta\phi$. Therefore,

$$\begin{aligned} \langle\zeta(\vec{x})\zeta(\vec{y})\rangle &= \left(\frac{V'(\phi)}{\dot{\phi}^2}\right)^2 \langle\delta\phi(\vec{x})\delta\phi(\vec{y})\rangle \\ &= \left(\frac{V'(\phi)}{\dot{\phi}^2}\right)^2 \int_{H<p<He^{Ht}} \frac{d^3\vec{p}}{(2\pi)^3} e^{i\vec{p}\cdot(\vec{x}-\vec{y})} \frac{H^2}{2p^3}. \end{aligned} \tag{16}$$

Using the equation of motion during the slow-roll regime, $3H\dot{\phi} + V' = 0$ and $H^2 = \frac{8\pi}{3} G_N V = V/(3M_{Pl}^2)$, the prefactor is $V'/\dot{\phi}^2 = 9H^2/V'$,

$$\langle\zeta(\vec{x})\zeta(\vec{y})\rangle = \left(\frac{9H^3}{V'}\right)^2 \int_{H<p<He^{Ht}} \frac{d^3\vec{p}}{(2\pi)^3} e^{i\vec{p}\cdot(\vec{x}-\vec{y})} \frac{1}{2p^3} \, . \tag{17}$$

When this quantity comes back into horizon, well after the inflation is over, it is

$$\zeta = \frac{\delta\rho}{\rho + p} = \begin{cases} \frac{3}{4}\frac{\delta\rho}{\rho} & \text{radiation dominant} \\ \frac{\delta\rho}{\rho} & \text{matter dominant} \end{cases} \tag{18}$$

This way, the density fluctuation for the mode when it enters the horizon is given in terms of what was generated at the time of the inflation.

Of course we do not observe the density fluctuation as its mode entered the horizon; it has evolved and grew since. This is the topic of structure formation we did not cover in this course. Again the book by Dodelson is a good reference on this question. What is striking is the fact that the prediction of inflationary cosmology, coupled to the growth of structure once the mode enteres the horizon, is in an excellent agreement with what is observed in the galaxy-galaxy correlation function called power spectrum.

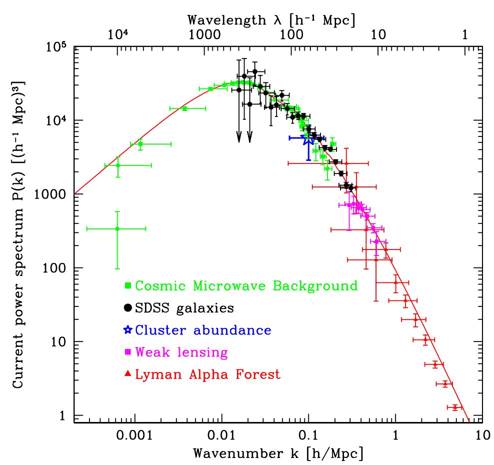

Figure 1: The agreement of the prediction by inflation and the observed power spectrum, Fig. 37 in M. Tegmark et al, *Astrop. J.*, **606**, 702 (2004).