Faddeev–Popov ghosts

Physics 230A, Spring 2007, Hitoshi Murayama

In this lecture notes, I describe what I discussed in class about Faddeev–Popov ghosts except for the material already described in Peskin–Schroeder.

1 Why Faddeev–Popov Determinant?

Suppose you have an integral whose integrand is highly symmetric. It is a kind of waste to integrate over the entire volume when you know you are adding the same integrand over and over again. You then would want to reduce the integration volume using the symmetry of the integrand. This is where the Faddeev–Popov method comes in.

Let us look at this simple example. You have an integral of a rotationally symmetric function $f(r, \alpha)$ which depends only on the radius $r = |\vec{x}|$ and other "physical" parameters α. You start with a two-dimensional integral

$$Z(\alpha) = \int d^2x f(r, \alpha). \tag{1}$$

In the end, we are only interested in the α dependence of the result, not the overall normalization. In this case, it is obvious what we should do. We change the integration variables to the polar coordinates and throw away the angular integral,

$$Z(\alpha) = \int_0^\infty r dr \int_0^{2\pi} d\phi f(r, \alpha) = 2\pi \int_0^\infty r dr f(r, \alpha) \propto \int_0^\infty r dr f(r, \alpha). \tag{2}$$

Now we do the same exercise in a different way. In many cases, it is not easy to find the change of variables to eliminate redundant variables explicitly. It would be nice to have a method that would apply more generally even when you don't know what the good variables are.

If I am naive, I may do the following which leads to a wrong result. I argue that the integrand depends on the radius alone, and it shouldn't matter along what angle ϕ I integrate. Then I can decide to integrate only along the x-axis and still get the same answer up to an overall constant. I would write

$$Z(\alpha) \propto \int_0^\infty dx f(x, \alpha). \tag{3}$$

We know this is wrong. Where did I make a mistake?

When I restricted the integral to the x-axis alone, I basically inserted $\delta(y)$ into the integrand. The problem is that this delta function does not behave "nicely" under rotation. To see this, let us check what happens when I rotate the coordinates from y to $y_\theta = y\cos\theta - x\sin\theta$:

$$\delta(y) \to \delta(y\cos\theta - x\sin\theta) = \delta(\theta - \tan^{-1}\frac{y}{x})\frac{1}{|y\sin\theta - x\cos\theta|} . \tag{4}$$

Namely it does not fix the angle θ without an additional θ dependent factor.

What we want is to just fix the angular integral, namely $\delta(\theta)$, not $\delta(y)$. We know this because we can separate variables easily in this simple case. In more complicated cases, we don't necessarily know. What we can do is to start with $\delta(y)$, but then compensate for the rotation we did above as:

$$\begin{aligned} 1 &= \int_0^\pi d\theta\delta(\theta - \tan^{-1}\frac{y}{x}) \\ &= \int_0^\pi d\theta\delta(y\cos\theta - x\sin\theta)|y\sin\theta - x\cos\theta| = \int_0^\pi d\theta\delta(y_\theta)\left|\frac{\partial y_\theta}{\partial\theta}\right| . \end{aligned} \tag{5}$$

We insert this expression into the original integral, and we find

$$Z(\alpha) = \int d^2x \int_0^\pi d\theta\delta(y_\theta)\left|\frac{\partial y_\theta}{\partial\theta}\right| f(r,\alpha). \tag{6}$$

We can now change the integration variable from y to y_θ so that it becomes

$$Z(\alpha) = \int d^2x \int_0^\pi d\theta\delta(y)\left|\frac{\partial y_\theta}{\partial\theta}\right|_{\theta=0} f(r,\alpha). \tag{7}$$

Now there is no θ dependence in the integrand anymore, so that it just gives a factor of π. We also know

$$\left|\frac{\partial y_\theta}{\partial\theta}\right|_{\theta=0} = |y\sin\theta - x\cos\theta|_{\theta=0} = |x|. \tag{8}$$

Therefore,

$$Z(\alpha) = \pi\int d^2x\delta(y)|x|f(r,\alpha) = \pi\int_{-\infty}^{\infty} dx|x|f(r,\alpha) = 2\pi\int_0^\infty dx x f(x,\alpha). \tag{9}$$

This is exactly what we want. Faddeev–Popov method is nothing but the generalization of what we have done here.

Let us generalize the above example to D-dimensional integral. For a rotationally invariant integrand $f(r, \alpha)$, we integrate over the entire D-dimensional volume

$$Z(\alpha) = \int d^D x f(r, \alpha). \tag{10}$$

We would like to impose the "gauge fixing condition" $x^2 = x^3 = \cdots = x^D = 0$ and keep only x^1 as the integration variable. There are $D - 1$ rotations that change the gauge fixing condition on the $(1, i)$ plane,

$$x^1 \to x^1 \cos\theta_{1i} - x^i \sin\theta_{1i}, \tag{11}$$

$$x^i \to x^1 \sin\theta_{1i} + x^i \cos\theta_{1i}. \tag{12}$$

Following the Faddeev–Popov method, we insert

$$1 = \int d\theta_{1i} \delta(x^1 \sin\theta_{1i} + x^i \cos\theta_{1i}) |x^1 \cos\theta_{1i} - x^i \sin\theta_{1i}| = \int d\theta_{1i} \delta(x^i_{\theta_{1i}}) \left| \frac{dx^i_{\theta_{1i}}}{d\theta_{1i}} \right|, \tag{13}$$

for each $i = 2, \cdots, D$. Then the original integral becomes

$$Z(\alpha) = \int d^D x f(r, \alpha) \prod_{i=2}^{D} d\theta_{1i} \delta(x^i_{\theta_{1i}}) \left| \frac{dx^i_{\theta_{1i}}}{d\theta_{1i}} \right|. \tag{14}$$

Rotate each of $x^i_{\theta_{1i}}$ back to x^i, we find

$$Z(\alpha) = \int d^D x f(r, \alpha) \prod_{i=2}^{D} d\theta_{1i} \delta(x^i) \left| x^1 \right| \propto \int dx^1 f(r, \alpha) |x^1|^{D-1}. \tag{15}$$

We know this is the correct result because we could have gone to spherical coordinates from the beginning. The final form still "double counts" the integrand because x^1 is integrated from $-\infty$ to ∞, but it is much more efficient than the original full volume integral. Note that we didn't have to think hard about the integration regions for the angles θ_{1i} because we were not concerned with the overall normalization. Namely that we only needed information about infinitesimal symmetry transformations, which is the beauty of the Faddeev–Popov method.

More to come.