129B Solutions to HW#2

- **1.** $g^2 = 8\, m_W^2\, G_F / \sqrt{2} = 8 * 80.33^2 * 1.1664 * 10^{-5} / \sqrt{2} = 0.4258 \longrightarrow g = 0.6523.$

  The main source of error is the uncertainty in the W-boson mass (±0.15 GeV), the Fermi constant is measured to 6 decimal places. $g \propto m_W$, hence $\sigma(g) = \sigma(m_W) * (g / m_W) = 1.2 * 10^{-3}$.

- **2.** $g^2 / (4\pi) = 0.426 / (4\pi) = 3.39 * 10^{-2}$. **This number is larger than the QED fine structure constant $\alpha = e^2 / (4\pi) = 1/137 = 0.730 * 10^{-2}$. The weak interaction is actually *stronger* than electromagnetic at very small distances. Of course, the reason why we call it *weak* is because the W-boson that mediates it is very heavy and hence the range of the interaction is very small.**

- **3. Using the following formula for the decay rate (derived later in the Optional),**

$$\Gamma(W^- \longrightarrow e^- \bar{\nu}_e) = \frac{g^2}{(48\pi)} m_W, \tag{1}$$

we find $\Gamma_{predicted} = 0.426 * 80.33 / (48\pi) = 0.227\ \text{GeV}.$

The measured value is $\Gamma_{measured} = 2.07 * 0.108 = 0.227\ \text{GeV}.$

The error bar for $\Gamma_{measured}$ **is equal to** $0.224 * \sqrt{\left(\frac{0.4}{10.8}\right)^2 + \left(\frac{0.06}{2.07}\right)^2} = 0.01\ \text{GeV}$, **so the two numbers are completely consistent.**

Optional

Here you are asked to derive the expression (1) for the "partial" decay rate of the W-boson into $e^- \bar{\nu}_e$ from the first principles (*i. e.*, Feynman Rules). The starting point is the Feynman amplitude:

$$i M = \frac{i g}{\sqrt{2}} \bar{u}(p_e) \gamma_\mu \frac{1 - \gamma_5}{2} v(p_\nu) \epsilon^\mu (p_W), \tag{2}$$

where $\epsilon^\mu(p_W)$ is chosen to be $\frac{1}{\sqrt{2}}(0, 1, i, 0)$, i. e. the spin of the W-boson is pointing in the positive z-direction; $u(p_e)$ is the wavefunction of the electron going in the $(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ direction and $v(p_\nu)$ is the wavefunction of the antineutrino going in the *opposite* direction.

The derivation consists of two steps:

• First, plug in the expressions for the wavefunctions, and multiply through the matrices. This answers part b). Consult the handout on the solutions to the Dirac equation from the last semester for the exact expressions for $u(p_e)$ and $v(p_\nu)$.

• Next, plug in the result for $|M|^2$ into the expression for the decay rate (Eq (1) from the handout on cross sections, decay rates, etc.) and perform the phase space integration. This is all there is to part a).

In the following we neglect the mass of the electron. This is justified, since its energy in the process $(= m_W/2 \simeq 40\ \text{GeV})$ is much greater that its rest mass.

With this assumption the two possible helicity states are:

$u_+ = \sqrt{E}\begin{pmatrix}\chi_+\\ \chi_+\end{pmatrix}$ and $u_- = \sqrt{E}\begin{pmatrix}\chi_-\\ -\chi_-\end{pmatrix}$. We will demonstrate that only u_- state participates in this interaction.

For a massless antiparticle the corresponding states are:

$v_+ = \sqrt{E}\begin{pmatrix}-\chi_-\\ \chi_-\end{pmatrix}$ and $v_- = \sqrt{E}\begin{pmatrix}\chi_+\\ \chi_+\end{pmatrix}$. An antineutrino can only be right-handed, i.e. $v(p_\nu) = v_+$.

Consider the action of $(1-\gamma_5)/2$ on the spinors. If this operator is applied on the right, it gives $(1-\gamma_5)/2 \cdot v_+ =$

$\frac{1}{2}\begin{pmatrix}I & -I\\ -I & I\end{pmatrix}\sqrt{E}\begin{pmatrix}-\chi_-\\ \chi_-\end{pmatrix} = \sqrt{E}\begin{pmatrix}-\chi_-\\ \chi_-\end{pmatrix} = v_+$. Notice, however, that if the antineutrino were instead left-handed, the result would be *zero*. We see that $(1-\gamma_5)/2$, when acts on the left, *projects out* the left-handed helicity state, because in that state 2 χ spinors enter with the same sign. Conclusion: even if the antineutrino had a left-handed component, that component would not couple to the *W*. What about the electron? It does have two components, but again only one (left-handed) couples to the *W*-boson. To see that, operate $(1-\gamma_5)/2$ on the left:

$\bar{u}(p_e)\gamma_\mu \frac{(1-\gamma_5)}{2} = u^\dagger(p_e)\gamma_0 \frac{1+\gamma_5}{2}\gamma_\mu = \left[u^\dagger(p_e)\frac{(1-\gamma_5)}{2}\right]\gamma_0\gamma_\mu$.

The bracketed expression vanishes when $u^\dagger = u_+^\dagger$, because in this case again there is a relative "+" sign between the two χ spinors.

To summarize, we have shown that

$$\frac{i g}{\sqrt{2}}\,\bar{u}(p_e)\gamma_\mu\frac{1-\gamma_5}{2}v(p_\nu) = \frac{i g}{\sqrt{2}}\,\bar{u}_-(p_e)\gamma_\mu v_+(p_\nu).$$

Now it's time to use the explicit form of the spinors $\chi_- = \begin{pmatrix} -\sin\frac{\theta}{2}\, e^{-i\phi} \\ \cos\frac{\theta}{2} \end{pmatrix}$**:**

$$\frac{i g}{\sqrt{2}}\, \bar{u}_-(p_e)\,(\gamma_\mu\, \epsilon_+^\mu)\, v_+(p_\nu) = \frac{i g}{\sqrt{2}} \sqrt{E_e} \times$$

$$\left(-\sin\frac{\theta}{2}\, e^{-i\phi},\ \cos\frac{\theta}{2},\ \sin\frac{\theta}{2}\, e^{-i\phi},\ -\cos\frac{\theta}{2}\right)^* \gamma_0\, (\gamma_\mu\, \epsilon_+^\mu) \sqrt{E_\nu} \begin{pmatrix} \sin\frac{\theta-\pi}{2}\, e^{-i\phi} \\ -\cos\frac{\theta-\pi}{2} \\ -\sin\frac{\theta-\pi}{2}\, e^{-i\phi} \\ \cos\frac{\theta-\pi}{2} \end{pmatrix} =$$

$$\frac{i g}{\sqrt{2}} \sqrt{E_e E_\nu} \left(-\sin\frac{\theta}{2}\, e^{i\phi},\ \cos\frac{\theta}{2},\ \sin\frac{\theta}{2}\, e^{i\phi},\ -\cos\frac{\theta}{2}\right) \times \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \times$$

$$\left(\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \right) \begin{pmatrix} -\cos\frac{\theta}{2}\, e^{-i\phi} \\ -\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2}\, e^{-i\phi} \\ \sin\frac{\theta}{2} \end{pmatrix} = \frac{i g}{2} \sqrt{E_e E_\nu} \times$$

$$\left(-\sin\frac{\theta}{2}\, e^{i\phi},\ \cos\frac{\theta}{2},\ \sin\frac{\theta}{2}\, e^{i\phi},\ -\cos\frac{\theta}{2}\right) \begin{pmatrix} 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -\cos\frac{\theta}{2}\, e^{-i\phi} \\ -\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2}\, e^{-i\phi} \\ \sin\frac{\theta}{2} \end{pmatrix} =$$

$$\frac{i g}{2} \sqrt{E_e E_\nu}\, 4 \sin^2\frac{\theta}{2}\, e^{i\phi} = i g \sqrt{E_e E_\nu}\, (1 - \cos\theta)\, e^{i\phi} = i g \frac{m_W}{2} (1 - \cos\theta)\, e^{i\phi}$$

This is the anticipated answer for part b). Notice the choice of the angle variable in v_+**: it is** $(\theta-\pi)/2$**, because the antineutrino travels in the opposite direction with repect to the electron (to which** θ **refers). Also notice that in the last step the fact that** $E_e = E_\nu = m_W / 2$ **was used.**

Let's discuss the θ**-dependence of the amplitude. As we concluded before, the spin of the electron is pointing in the direction opposite to its momentum and the spin of the antineutrino is pointing in the direction of its momentum. Therefore, the final state has angular momentum -1 with respect to** $\vec{p}_e$**, while the initial state angular momentum points along the** z**-axis. The problem, thus, reduces to a standard quantum mechanics question: if you prepare a state with angular momentum +1 along the** z**-direction, what is the amplitude of measuring angular momentum -1 along some other axis that makes an angle** θ **with the**

z-direction? The answer is given by the so-called d-functions, which can be found, for example, in the PDG next to the tables of Clebch-Gordan coefficients. In our particular case we start with angular momentum +1 (hence $d^1_{\square,\square}$), and end up with angular momentum 1 (hence $d^1_{1,\square}$) and projection -1 (hence $d^1_{1,-1}$). The PDG quotes $d^1_{1,-1} = (1-\cos\theta)/2$, just as we found! Of course, the amplitude should be zero if the electron in going in the positive z-direction, as it is impossible to conserve angular momentum in this case.

Now let's do the phase space integral:

$$\Gamma = \frac{1}{2M_W}\int |M|^2\, d\Phi =$$

$$\frac{1}{2M_W}\,\frac{g^2 M_W^2}{4}\int (1-\cos\theta)^2 (2\pi)^4 \delta^{(4)}(P_W - p_\nu - p_e)\,\frac{d^3\vec{p}_e}{(2\pi)^3 (2E_e)}\,\frac{d^3\vec{p}_\nu}{(2\pi)^3 (2E_\nu)} =$$

$$\frac{g^2 M_W}{(2\pi)^2\, 8}\int (1-\cos\theta)^2 \delta^{(4)}(E_W - E_\nu - E_e)\,\frac{d^3\vec{p}_e}{(2E_e)}\,\frac{1}{(2E_\nu)} = \ldots$$

Here $E_\nu = p_\nu = p_e = E_e$ (see the solution for HW #1 for details).

$$\ldots = \frac{g^2 M_W}{(2\pi)^2\, 8}\int (1-\cos\theta)^2 \delta^{(4)}(E_W - 2E_e)\,\frac{E_e^2\, dE_e\, d\Omega}{(2E_e)}\,\frac{1}{(2E_e)} =$$

$$\frac{g^2 M_W}{(2\pi)^2\, 32}\int (1-\cos\theta)^2 \delta^{(4)}(E_W - 2E_e)\, dE_e\, d\phi\, \sin\theta\, d\theta =$$

$$\frac{g^2 M_W}{(2\pi)^2\, 32}\int_0^{\pi} (1-\cos\theta)^2\,\frac{1}{2}\, d(-\cos\theta) \times \int_0^{2\pi} d\phi = \frac{g^2 M_W\, \pi}{(2\pi)^2\, 32}\times\left(\frac{2^3}{3}\right) = \frac{g^2 M_W}{48\pi}.$$